Varieties and Clones of Relational Structures

DISSERTATION

zur Erlangung des akademischen Grades

Doctor rerum naturalium
(Dr. rer. nat.)

vorgelegt

der Fakultät Mathematik und Naturwissenschaften
der Technischen Universität Dresden

von

Diplommathematiker Grabowski, Jens-Uwe

geboren am 6. Oktober 1968 in Dresden

Gutachter: Prof. Dr. Reinhard Pöschel
Prof. Dr. Ivo G. Rosenberg
Doz. Dr. László Zádori

Eingereicht am: 9. Januar 2002

Tag der Disputation: 7. Juni 2002

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Jens-Uwe Grabowski

July 18, 2002

Acknowledgments

First of all, I thank my supervisor Professor Reinhard Pöschel. He has guided and supported me not only during this PhD project, but also in the years before. I have profited much from his rich mathematical intuition.

I thank also my colleagues. Our discussions resulted in a lot of hints, ideas and motivation. In particular, I owe thanks to Rosemarie Nitschke and Uwe Iben for proofreading.

Finally, I want to mention my family, Beate and Astrid. Their encouragement and patience contributed much to my work.

Contents

Introduction

The aim of this PhD thesis is to develop further the theory of relational structures. Especially, we examine those properties that are related to the concepts of clones and varieties and that have connections with the field of universal algebra.

Three concepts are fundamental for our considerations: varieties and clones of relational structures, and relational equations. First, let us have a look at them.

The concept of a relational structure, i.e., a set together with a family of relations on this set, is one of the most general in mathematics. Several special types of relational structures are the objects of extensively developed theories. One example for this are ordered sets, which are widely applied in all branches of mathematics. The rising interest in general relational structures is caused by their applications, e.g. in theoretical computer science, as well as their connections with topics of universal algebra. As examples for the latter, we mention clone theory [Tay93, MMT87, Sze86, PK79], categorical equivalence of algebras [McK96, Zád97], and tame congruence theory [HM88, Kea01].

Varieties of relational structures, relational varieties for short, are classes of relational structures closed under formation of products and retracts. They have their origin in the theory of ordered sets. D. Duffus and I. Rival [DR81] have presented a discussion why relational varieties are particularly suited for a structure theory of ordered sets. To mention a classical result about order varieties, we refer to [RW81]. Both, product and retract, are concepts of category theory. While the product construction has received major attention in model theory, algebra, and other fields of mathematics, retracts of relational structures were not so widely applied [DR79, Riv82]. Many concepts and results for order varieties have been generalized to varieties of arbitrary relational structures, e.g. by L. Zádori [Zád97, Zád98].

Beside relational varieties, we consider two variants involving finiteness conditions. Pseudovarieties are classes of finite relational structures closed under formation of finite products and retracts. Finitely generated varieties are classes generated (with respect to formation of products and retracts) by finitely many finite structures. The main stream of considerations we make deals with finite structures and finite algebras. In many cases we extend results partially to the infinite case, or give examples why finiteness is essential.

A clone of functions on a set A is a set of functions on A closed under superposition and containing all projections. A natural example is the set of all term functions of an algebra. We call a relation to be invariant under a function f if it is a subuniverse of a finite power of the algebra with one base function f. This gives rise to a Galois correspondence between sets of functions and sets of relations. It turns out that, in the case of finite base sets, the Galois closed sets of functions are exactly the clones of functions. Further, the Galois closed sets of relations can be described in a way analogous to functions, namely, in terms of closure under so called primitive-positive constructions. A detailed survey of this field can be found in [PK79].

Having this at hand, we address the question whether the concepts of term and equation can be translated to relations. The considerations above suggest us using primitive-positive formulas as "relational terms" and defining a relational equation to be an expression of the form $\varphi_1 \leftrightarrow \varphi_2$, where φ_1 and φ_2 are primitive-positive formulas. For example, $((\exists y)\ x \leq y \leq z) \leftrightarrow (x \leq z)$ is a relational equation stating the transitivity of $\leq$. Our main result in Chapter 2 shows that this approach reaches far: Finitely generated pseudovarieties can be axiomatized by (or are the model classes of) sets of relational equations. This gives a model-theoretic approach to relational varieties. Note the analogy with algebras, where the famous result of G. Birkhoff states that varieties of algebras, i.e., classes of algebras closed under formation of products, homomorphic images and subalgebras, are exactly the classes axiomatized by sets of equations.

Since our approach to relational equations originates in a duality between functions and relations, one expects that results concerning relational equations have counterparts in the world of functions. In Chapter 4 we examine this. It turns out that products and retracts correspond to certain constructions of algebras, and that structures satisfying the same relational equations correspond to algebras which are categorically equivalent. Two algebras $\mathbb{A}$ and $\mathbb{B}$ are called categorically equivalent if there is an equivalence functor between the variety generated by $\mathbb{A}$ and the variety generated by $\mathbb{B}$ (construed as categories) mapping $\mathbb{A}$ to $\mathbb{B}$. Categorical equivalence originally appeared in the study of special classes of algebras, e.g. modules (Morita), primal algebras (Hu), or unary algebras. Perhaps, the most prominent example in universal algebra is Hu's Theorem [Hu69], which states that for any primal algebra $\mathbb{A}$ the class of all algebras categorically equivalent to $\mathbb{A}$ is the class of all primal algebras. A consequence is that primal algebras share interesting properties with the two-element Boolean algebra, e.g. a variety generated by a primal algebra can be represented in an especially simple form. Beside primality, a long list of properties of algebras are preserved under categorical equivalence, see e.g. [McK96, BB96] and the references cited therein.

This work is organized as follows:

Chapter 1. We introduce the basic definitions regarding structures, clones and formulas.

Chapter 2. We consider basic constructions of relational structures and how they interact with certain classes of formulas. Especially, we introduce relational equations, compare them with Horn-sentences, and we see that validity of relational equations is preserved under formation of products, retracts and direct limits of direct families. In Section 2.3 we prove the main result, mentioned above, that if K is a finite class of finite structures and $\mathbf{A}$ is a finite structure satisfying all relational equations valid in K then $\mathbf{A}$ lies in the pseudovariety generated by K. We discuss some variants of the finiteness conditions. In the final section of this chapter we modify this result and show how finite classes of finite structures closed under formation of retracts can be axiomatized.

Chapter 3. We present a different approach to define relational clones. This enables us to construe relational clones as algebras and to employ facts and notions from universal algebra in our considerations about relational equations. Especially, we define the notions of isomorphisms and homomorphisms between relational clones and examine the consequences for the generating structures of these clones. This leads us to connections of relational equations with the lattice of all clones.

Chapter 4. We connect relational structures and algebras in a natural way, namely, we relate an algebra $\mathbb{A}$ with a structure $\mathbf{A}$ if the clone of $\mathbf{A}$ is the set of invariant relations of $\mathbb{A}$, and recover the following pairs of corresponding notions:

$$\begin{array}{rl} \text{retract} & \text{idempotent image,} \\ \text{power} & \text{matrix power,} \\ \mathrm{Th}_{\mathrm{re}}\text{-equivalent} & \text{categorically equivalent,} \end{array}$$

where $\mathrm{Th}_{\mathrm{re}}$-equivalent stands for the fact that two relational structures satisfy the same relational equations. We obtain an algebraic counterpart of the result in Section 2.3. Further, we utilize the connections between relational structures and algebras to examine how some properties of algebras, which are determined by their invariant relations, propagate under categorical equivalence.

Chapter 5. For two special classes of structures, namely structures with minimal clones and two-element (Boolean) structures, we discuss a classification by relational equations. This provides a large collection of examples for relational varieties and the concepts developed in Chapter 2. The topic of Boolean structures is closely related to several generalizations of lattices and Boolean algebras, e.g. ordered sets with complements. In section Section 5.2 we show that a relational variety generated by a rigid structure $\mathbf{B}$ can be described in an especially simple form, namely, it consists essentially of reduced powers of $\mathbf{B}$. This is analogous to the well known fact that a variety generated by a primal algebra is of a special form, namely, it consists of the Boolean powers of the generating algebra (Foster, see e.g. [BS81]).

The main results of Sections 2.3 and 4.2 have been published in [Gra01].

Chapter 1

Preliminaries

We introduce basic notions and notations. The presentation in Sections 1.2 and 1.3 concerning relational structures and formulas follows mainly [BS81]. In Section 1.4 we present some notions from the theory of clones of functions and clones of relations; a detailed introduction can be found in [PK79]. We assume a familiarity with the most basic concepts of set theory, universal algebra, and model theory, which can be found e.g. in [BS81], and just mention the notation used.

1.1 General notation

We denote by $\mathbb{N}$ the nonnegative integers $0, 1, \dots$, and by $\mathbb{N}_+$ the set of positive integers $1, 2, \dots$.

Let I, A and A_i, $i \in I$, be sets. We use the following set-theoretical notations: $\prod_I A_i$ (cartesian product), A^I (cartesian power), and $\mathcal{P}(A)$ (power set of A). Given $i^* \in I$ and $a \in \prod_I A_i$, we write $a(i^*)$ for the i^*-th component of a. The i^*-th projection map $\pi_{i^*} \colon \prod_I A_i \to A_{i^*}$ is defined by $\pi_{i^*}(a) := a(i^*)$. In the special case $I = \{0, \dots, m-1\}$, the members of $\prod_I A_i$ are ordered m-tuples and we display them like $\langle a_0, \dots, a_{m-1} \rangle$, and the cartesian power A^I is written as A^m.

Let $r \subseteq A^m$ be an m-ary relation on A. Instead of $\langle a_0, \dots, a_{m-1} \rangle \in r$ we often use the notations $r(a_0, \dots, a_{m-1})$ or, in the case $m = 2$, $a_0\ r\ a_1$ (infix notation).

Let $\alpha \colon A \to B$ and $\beta \colon B \to C$ be (unary) functions. The composition $\beta\alpha \colon A \to C$ of α and β is given by $(\beta\alpha)(a) := \beta(\alpha(a))$ for $a \in A$. By id_A we denote the identical mapping on A. We extend α in a usual way and define

$$\begin{aligned} \alpha(\langle a_0, \dots, a_{m-1} \rangle) &:= \langle \alpha(a_0), \dots, \alpha(a_{m-1}) \rangle, \qquad && \langle a_0, \dots, a_{m-1} \rangle \in A^m, \\ \alpha(A') &:= \{\alpha(a) \mid a \in A'\}, && A' \subseteq A, \end{aligned}$$

i.e., tuples are mapped "component-wise" and sets are mapped "element-wise", and

$$\alpha^{-1}(B') := \{a \in A \mid \alpha(a) \in B'\}, \qquad B' \subseteq B.$$

From this, we derive the following notations for relations $r \subseteq A^m$ and $r' \subseteq B^m$:

$$\alpha(r) = \{\langle \alpha(a_0), \dots, \alpha(a_{m-1})\rangle \mid \langle a_0, \dots, a_{m-1}\rangle \in r\},$$
$$\alpha^{-1}(r') = \{\langle a_0, \dots, a_{m-1}\rangle \mid \langle \alpha(a_0), \dots, \alpha(a_{m-1})\rangle \in r'\}.$$

To distinguish functions of different role, we use lower-case Greek letters $\alpha, \beta, \dots$ especially for mappings between base sets of structures and algebras, while functions occurring as base functions or term functions of algebras are denoted by $f, g, \dots$. The following notation is applied to the latter type of functions. Let $n, m \in \mathbb{N}_+$ and let A be a set. We define

$$\begin{aligned}
\mathrm{Func}^{(n)}(A) &:= \{f \mid f\colon A^n \to A\}, && \text{the set of all } n\text{-ary functions on } A,\\
\mathrm{Func}(A) &:= \bigcup_{n=1}^{\infty} \mathrm{Func}^{(n)}(A), && \text{the set of all finitary functions on } A,\\
\mathrm{Rel}^{(m)}(A) &:= \{r \mid r \subseteq A^m\}, && \text{the set of all } m\text{-ary relations on } A,\\
\mathrm{Rel}(A) &:= \bigcup_{m=1}^{\infty} \mathrm{Rel}^{(m)}(A), && \text{the set of all finitary relations on } A.
\end{aligned}$$

By $\mathrm{Func}^{(1-1)}(A)$ we denote the set of all unary, bijective functions on A. Let $A' \subseteq A$ and $f \in \mathrm{Func}(A)$. We denote by $f\restriction_{A'}$ the restriction of f to A'.

For $F \subseteq \mathrm{Func}(A)$ and $R \subseteq \mathrm{Rel}(A)$ we define

$$\begin{aligned}
F^{(n)} &:= F \cap \mathrm{Func}^{(n)}(A), && \text{the } n\text{-ary functions of } F,\\
R^{(m)} &:= R \cap \mathrm{Rel}^{(m)}(A), && \text{the } m\text{-ary relations of } R.
\end{aligned}$$

1.2 Relational structures

Definition 1.1. A *relational type* (or *type*) $\mathcal{R}$ is a first-order type without function symbols, i.e., $\mathcal{R}$ is a set of relation symbols with an arity $\mathrm{ar}(\mathsf{r}) \in \mathbb{N}_+$ assigned to each relation symbol $\mathsf{r} \in \mathcal{R}$.

Definition 1.2. Let $\mathcal{R}$ be a relational type. A *relational structure* (or *structure*) $\mathbf{A}$ of type $\mathcal{R}$ is a pair (A, R), where A is a nonempty set and $R = (\mathsf{r}^{\mathbf{A}} \mid \mathsf{r} \in \mathcal{R})$ is a family of relations on A indexed by $\mathcal{R}$. For $\mathsf{r} \in \mathcal{R}$, the relation $\mathsf{r}^{\mathbf{A}}$ has arity $\mathrm{ar}(\mathsf{r})$, i.e., $\mathsf{r}^{\mathbf{A}} \subseteq A^{\mathrm{ar}(\mathsf{r})}$. We call A the *base set* of $\mathbf{A}$, and R the *base relations* of $\mathbf{A}$. We call $\mathbf{A}$ *finite* if A is finite.

We follow the usual convention to denote a structure by the same letter as the base set. We fix a type $\mathcal{R}$ and consider all structures to belong to $\mathcal{R}$ unless stated otherwise. When we refer to a property of a mapping between base sets of structures we write $\alpha\colon \mathbf{A} \to \mathbf{B}$ instead of $\alpha\colon A \to B$ and mention the property explicitly, e.g. "$\alpha\colon \mathbf{A} \to \mathbf{B}$ is a homomorphism".

To distinguish algebras from structures clearly, we denote structures by $\mathbf{A}, \mathbf{B}, \dots$, and algebras by $\mathbb{A}, \mathbb{B}, \dots$.

1.3 Formulas

We fix a set $X = \{x_0, x_1, \dots\}$ of variables. Let $\varphi(x_0, \dots, x_{m-1})$ be a first-order formula of type $\mathcal{R}$. As usual, we express by this notation that at most

$x_0, \ldots, x_{m-1}$ occur free in φ. Moreover, this notation assigns a unique arity $\mathrm{ar}(\varphi) = m$ to φ. By $\mathbf{A} \models \varphi(a_0, \ldots, a_{m-1})$ we denote that $\varphi(a_0, \ldots, a_{m-1})$ holds in $\mathbf{A}$. Further, we use the following model-theoretical notations. Let $\mathbf{A}$ be a structure, K a class of structures, φ a sentence, and Σ, Σ' sets of sentences.

$$\begin{aligned}
\mathbf{A} &\models \varphi, && (\varphi \text{ holds in } \mathbf{A})\\
\mathbf{A} &\models \Sigma, && (\text{all } \varphi \in \Sigma \text{ hold in } \mathbf{A})\\
K &\models \varphi, && (\varphi \text{ holds in all } \mathbf{A} \in K)\\
K &\models \Sigma, && (\text{all } \varphi \in \Sigma \text{ hold in all } \mathbf{A} \in K)\\
\Sigma &\models \Sigma', && (\Sigma \text{ yields } \Sigma')
\end{aligned}$$

We denote by $\mathrm{Mod}\,\Sigma$ the class of all structures $\mathbf{A}$ satisfying $\mathbf{A} \models \Sigma$, and by $\mathrm{Th}\,K$ the set of all sentences φ satisfying $K \models \varphi$.

Definition 1.3. Let $\mathbf{A}$ be a structure and let $\varphi(x_0, \ldots, x_{m-1})$ be a first-order formula. Then $\varphi^{\mathbf{A}} \subseteq A^m$ is defined by

$$\varphi^{\mathbf{A}} := \{\langle a_0, \ldots, a_{m-1}\rangle \mid \mathbf{A} \models \varphi(a_0, \ldots, a_{m-1})\}.$$

So the statements and $\mathbf{A} \models \varphi(a_0, \ldots, a_{m-1})$ and $\langle a_0, \ldots, a_{m-1}\rangle \in \varphi^{\mathbf{A}}$ are equivalent. We use the second form when we want to put emphasis on that $\varphi^{\mathbf{A}}$ is a set of tuples.

Beside the quantifiers ($\exists$, $\forall$), junctors ($\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftarrow$, $\leftrightarrow$) and equality ($\approx$), we use the atomic formulas f and t which are interpreted as "always false" and "always true" respectively. Note that f and t can be expressed by the other symbols, they are provided for the sake of brevity.

Given a finite set I and formulas φ_i, $i \in I$, we denote the conjunction and disjunction of the φ_i by $\bigwedge_{i\in I} \varphi_i$ and $\bigvee_{i \in I} \varphi_i$ respectively. To denote any conjunction (disjunction) of finitely many formulas, we often write shorter $\bigwedge_i \varphi_i$ and $\bigvee_i \varphi_i$.

We want to denote several classes of formulas according to the occurring junctors and quantifiers.

Definition 1.4. Let $S \subseteq \{\exists, \forall, \wedge, \vee, \neg, \approx, \mathsf{f}, \mathsf{t}\}$. Then $\Phi(S)$ denotes the set of all first-order formulas of arity at least one, which contain, beside variables and relation symbols, only symbols from S. We call $\Phi(\exists, \wedge, \approx, \mathsf{f}, \mathsf{t})$ the set of *primitive-positive formulas*, and we call $\Phi(\exists, \wedge, \vee, \approx, \mathsf{f}, \mathsf{t})$ the set of *existential-positive formulas.*

For example, $\Phi(\forall, \exists, \wedge, \vee, \approx, \mathsf{t})$ is known as the set of positive formulas. If a symbol in S can be expressed by the others, $\Phi(S)$ remains unchanged up to equivalence of formulas when we omit the symbol from S. For instance, $\Phi(\wedge, \vee, \neg)$ is equivalent to $\Phi(\wedge, \neg)$. When writing $\Phi(S)$, we include all possible symbols, but keep in mind that we can deal with a reduced S, for instance in proofs that use induction on formulas.

We exclude formulas of arity zero from $\Phi(S)$ to ensure that $\varphi^{\mathbf{A}}$ is always a relation for $\varphi \in \Phi(S)$. Note that $\varphi(x_0, \ldots, x_{m-1})$ is primitive-positive if and only if it is $\mathsf{f}(x_0, \ldots, x_{m-1})$ or it is equivalent to a formula of the form

$$(\exists x_m, \ldots, x_{l-1}) \ \bigwedge\nolimits_j \mathsf{r}_j(\ldots) \wedge \bigwedge\nolimits_k x_{1k} \approx x_{2k},$$

where the r_j are in $\mathcal{R}$, the variables occurring in the r_j are in $\{x_0, \dots, x_{l-1}\}$, and the variables x_{1k} and x_{2k} are in $\{x_0, \dots, x_{m-1}\}$.

We often abbreviate finite tuples like $a_0, \dots, a_{m-1}$ by $\bar{a}$, especially when $\bar{a}$ is a member of a relation as in $r(\bar{a})$ or $\bar{a} \in r$. We do not use this notation to shorten the argument tuple of an n-ary function as in $f(a_0, \dots, a_{n-1})$, since this could cause ambiguity with a *unary* function mapping a tuple component-wise as defined in Section 1.1. In the same manner, we use $\bar{x}$ for tuples of variables occurring in formulas, e.g. we write $(\exists \bar{x})$ or $\varphi(\bar{x})$.

1.4 Polymorphisms and invariant relations

Polymorphisms and invariant relations provide a connection between functions and relations. Moreover, they are closely related with clones of functions and clones of relations. Especially, clones of functions enjoy a deeply developed theory [Tay93] and play an important role in universal algebra, see e.g. [MMT87, Sze86]. In Chapter 3 we present further facts from the theory of algebras of functions and algebras of relations. We follow mainly the notation given in [PK79].

Definition 1.5. Let $n, m \in \mathbb{N}_+$. A function $f \in \mathrm{Func}^{(n)}(A)$ *preserves* a relation $r \in \mathrm{Rel}^{(m)}(A)$ if for all $\bar{a}_0, \dots, \bar{a}_{n-1} \in r$ we have

$$f(\bar{a}_0, \dots, \bar{a}_{n-1}) \in r,$$

where $f(\bar{a}_0, \dots, \bar{a}_{n-1})$ is defined component-wise, i.e.,

$$f(\bar{a}_0, \dots, \bar{a}_{n-1})(i) := f(\bar{a}_0(i), \dots, \bar{a}_{n-1}(i)), \qquad i \in \{0, \dots, m-1\}.$$

In this case, we call f a *polymorphism* of r, and r an *invariant relation* of f. For $F \subseteq \mathrm{Func}(A)$ and $R \subseteq \mathrm{Rel}(A)$ we denote

$$\begin{aligned} \mathrm{Inv}_A F &:= \{r \in \mathrm{Rel}(A) \mid (\forall f \in F)\ f \text{ preserves } r\}, \\ \mathrm{Pol}_A R &:= \{f \in \mathrm{Func}(A) \mid (\forall r \in R)\ f \text{ preserves } r\}. \end{aligned}$$

The *clone* $\mathrm{Cln}\,\mathbb{A}$ of an algebra $\mathbb{A}$ is the set of all term functions of $\mathbb{A}$. For $F \subseteq \mathrm{Func}(A)$ we set $\mathrm{Cln}_A F := \mathrm{Cln}(A, F)$, the *clone generated by* F. We call F a *clone* if $F = \mathrm{Cln}_A F$.

The *clone* $\mathrm{Cln}\,\mathbf{A}$ of a relational structure $\mathbf{A}$ is defined by

$$\mathrm{Cln}\,\mathbf{A} := \{\varphi^{\mathbf{A}} \mid \varphi \text{ is a primitive-positive formula}\},$$

and for $R \subseteq \mathrm{Rel}(A)$ we set $\mathrm{Cln}_A R := \mathrm{Cln}(A, R)$, the *clone generated by* R. We call R a *clone* if $R = \mathrm{Cln}_A R$. We use the name clone and the operator Cln for both, sets of functions and sets of relations. This is to emphasize the dual roles they play, and does not cause ambiguity. If A is clear, the subscript A in Inv_A,

Pol_A and Cln_A is omitted. We set

$$\begin{aligned}
\mathrm{Inv}^{(m)} F &:= \mathrm{Inv}\, F \cap \mathrm{Rel}^{(m)}(A), \\
\mathrm{Pol}^{(n)} R &:= \mathrm{Pol}\, R \cap \mathrm{Func}^{(n)}(A), \\
\mathrm{End}\, R &:= \mathrm{Pol}\, R \cap \mathrm{Func}^{(1)}(A), \\
\mathrm{Cln}^{(n)} F &:= \mathrm{Cln}\, F \cap \mathrm{Func}^{(n)}(A), \\
\mathrm{Cln}^{(m)} R &:= \mathrm{Cln}\, R \cap \mathrm{Rel}^{(m)}(A).
\end{aligned}$$

We extend these operators to algebras and structures, i.e., for an algebra $\mathbb{A} = (A, F)$ and a structure $\mathbf{A} = (A, R)$ we set

$$\begin{aligned}
\mathrm{Inv}\, \mathbb{A} &:= \mathrm{Inv}_A\, F, \\
\mathrm{Pol}\, \mathbf{A} &:= \mathrm{Pol}_A\, R, \\
\mathrm{End}\, \mathbf{A} &:= \mathrm{End}_A\, R.
\end{aligned}$$

For finite base sets A we define

$$\begin{aligned}
\mathrm{Aut}\, R &:= \mathrm{Pol}\, R \cap \mathrm{Func}^{(1-1)}(A), \\
\mathrm{Aut}\, \mathbf{A} &:= \mathrm{Aut}_A\, R.
\end{aligned}$$

To illustrate this definition, we express some known concepts in terms of it:

- $\mathrm{End}\, \mathbf{A}$ is the set of endomorphisms of the structure $\mathbf{A}$ in the usual sense,
- $\mathrm{Aut}\, \mathbf{A}$ is the set of automorphisms of the finite structure $\mathbf{A}$ in the usual sense,
- a function is monotone with respect to an order relation $\leq$ if and only if it preserves $\leq$,
- the equivalence relations in $\mathrm{Inv}\, \mathbb{A}$ are the congruence relations of the algebra $\mathbb{A}$,
- $\mathrm{Inv}^{(1)}\, \mathbb{A}$ are the base sets of subalgebras of the algebra $\mathbb{A}$.

An alternative way to define polymorphisms and invariant relations is as follows. A function $f \in \mathrm{Func}^{(n)}(A)$ is a polymorphism of the structure $\mathbf{A}$ if it is a homomorphism from $\mathbf{A}^n$ to $\mathbf{A}$ (we define powers and homomorphisms of structures in Definitions 2.1 and 2.3). A relation $r \in \mathrm{Rel}^{(m)}(A)$ is an invariant relation of the algebra $\mathbb{A}$ if it is the base set of a subalgebra of $\mathbb{A}^m$.

In the literature, the notation Pol is used in two different meanings. One is that of polymorphisms as defined above, the other is that of polynomials. The reader should be aware of this to avoid confusion. We do not use Pol to denote polynomials.

For a fixed base set A, the operator Cln_A on $\mathrm{Rel}(A)$ (on $\mathrm{Func}(A)$ resp.) is an algebraic closure operator. We denote the lattice of clones on $\mathrm{Rel}(A)$ (on $\mathrm{Func}(A)$ resp.) with respect to inclusion $\subseteq$ by $\mathbf{L}_A$ ($\mathbf{L}'_A$ resp.).

The operators Pol and Inv establish a Galois connection. This implies the following inclusions.

- For $F_1, F_2 \subseteq \mathrm{Func}(A)$ and $R_1, R_2 \subseteq \mathrm{Rel}(A)$,

$$\begin{aligned} F_1 \subseteq F_2 \quad &\text{implies} \quad \mathrm{Inv}\, F_1 \supseteq \mathrm{Inv}\, F_2, \\ R_1 \subseteq R_2 \quad &\text{implies} \quad \mathrm{Pol}\, R_1 \supseteq \mathrm{Pol}\, R_2. \end{aligned}$$

- For $F \subseteq \mathrm{Func}(A)$ and $R \subseteq \mathrm{Rel}(A)$,

$$\begin{aligned} R &\subseteq \mathrm{Inv}\,\mathrm{Pol}\, R, \\ F &\subseteq \mathrm{Pol}\,\mathrm{Inv}\, F, \\ \mathrm{Pol}\, R &= \mathrm{Pol}\,\mathrm{Inv}\,\mathrm{Pol}\, R, \\ \mathrm{Inv}\, F &= \mathrm{Inv}\,\mathrm{Pol}\,\mathrm{Inv}\, F. \end{aligned}$$

The same facts hold if Pol is replaced by End or Aut. For finite base sets A, the Galois closed sets of functions and relations are characterized in the following theorem.

Theorem 1.6 (see, e.g. [PK79]). *Let A be finite, let $F \subseteq \mathrm{Func}(A)$, and let $R \subseteq \mathrm{Rel}(A)$. Then*

$$\begin{aligned} \mathrm{Cln}\, R &= \mathrm{Inv}\,\mathrm{Pol}\, R, \\ \mathrm{Cln}\, F &= \mathrm{Pol}\,\mathrm{Inv}\, F. \end{aligned}$$

Hence, $\mathbf{L}_A$ and $\mathbf{L}'_A$ are anti-isomorphic. Further,

$$\begin{aligned} \mathrm{Inv}\,\mathrm{End}\, R &= \{\varphi^{(A,R)} \mid \varphi \textit{ is an existential-positive formula}\}, \\ \mathrm{Inv}\,\mathrm{Aut}\, R &= \{\varphi^{(A,R)} \mid \varphi \textit{ is a first-order formula}\}. \end{aligned}$$

In the literature, the closed sets of relations with respect to the Galois connection Inv – Aut (Inv – End resp.) characterized above are often called Krasner-clones (weak Krasner-clones resp.). The preceding theorem can be extended to the case of arbitrary base sets A, see e.g. [Pös79].

We finish this section with a look at the smallest clones and largest clones in $\mathbf{L}_A$ and $\mathbf{L}'_A$. Let A be fixed. Obviously, the largest clone of relations is $\mathrm{Rel}(A)$, and the largest clone of functions is $\mathrm{Func}(A)$.

The smallest clone of relations, denoted by $D(A)$, is $\mathrm{Cln}_A\, \emptyset$, that is, the set of all relations defined by a formula $\varphi(x_0, \dots, x_{m-1})$ which is $\mathsf{f}(x_0, \dots, x_{m-1})$ or which is of the form

$$\bigwedge\nolimits_k x_{1k} \approx x_{2k}.$$

We call a member of $D(A)$ a *diagonal relation*. The relations defined by f are the empty relations. For technical reasons, we consider empty relations of different arity to be distinct objects, and, if necessary, indicate the arity by writing $\emptyset^{(m)}$ for the m-ary empty relation.

The smallest clone of functions, denoted by $P(A)$, is $\mathrm{Cln}_A\, \emptyset$, that is, the set of all projections $\pi_i^{(n)} \colon A^n \to A$ for some $n \in \mathbb{N}_+$ and $i \in \{0, \dots, n-1\}$. Here, we include the arity n in the notation $\pi_i^{(n)}$ to easier distinguish projections of different arity.

For finite A, we conclude from Theorem 1.6

$$\begin{aligned}
\operatorname{Pol} D(A) &= \operatorname{Func}(A),\\
\operatorname{Inv} \operatorname{Func}(A) &= D(A),\\
\operatorname{Inv} P(A) &= \operatorname{Rel}(A),\\
\operatorname{Pol} \operatorname{Rel}(A) &= P(A).
\end{aligned}$$

Chapter 2

Constructions of relational structures

In Section 2.1 we define several constructions for relational structures, and examine how they interact. Three constructions—product, retract and direct limits of direct families—play a central role for our considerations.

In Section 2.2 we determine which classes $\Phi(S)$ of formulas are compatible with the constructions introduced in Section 2.1. In particular, we see that primitive-positive definitions are compatible with the formation of products and retracts of structures and with the formation of direct limits of direct families of structures.

Based on the rather technical results of Section 2.2, Section 2.3 is devoted to examine for which classes $\Sigma(S)$ of sentences (defined in Definition 2.19) validity of sentences from $\Sigma(S)$ is preserved under certain constructions. In particular, we see that validity of relational equations is preserved under products, retracts, and direct limits of direct families. We arrive at the main result of this chapter, Theorem 2.28, in showing that, under certain finiteness conditions, also the converse is true.

In Section 2.4 we modify Theorem 2.28 and present an axiomatization of finite classes of finite structures closed under retracts by positive equations.

2.1 Basic concepts

We define a sequence of constructions of relational structures—(reduced) products, (weak) substructures, (strong, full) homomorphisms, retracts, direct unions of directed families, and direct limits of direct families—and establish basic properties of them.

Definition 2.1. Let I be a nonempty set, and let $\mathbf{A}_i$, $i \in I$, be structures. The *product* $\prod_I \mathbf{A}_i$ is the structure whose base set is $\prod_I A_i$, and whose relations are defined component-wise for all $\mathsf{r} \in \mathcal{R}$:

$$\langle a_0, \ldots, a_{\mathrm{ar}(\mathsf{r})-1} \rangle \in \mathsf{r}^{\prod_I \mathbf{A}_i} \quad \text{iff} \quad \langle a_0(i), \ldots, a_{\mathrm{ar}(\mathsf{r})-1}(i) \rangle \in \mathsf{r}^{\mathbf{A}_i} \text{ for all } i \in I.$$

If all $\mathbf{A}_i$ are equal, i.e., $\mathbf{A}_i = \mathbf{A}$ for all $i \in I$, then $\prod_I \mathbf{A}_i$ is called a *power* of $\mathbf{A}$ and it is denoted by $\mathbf{A}^I$, or, in the special case $I = \{0, \ldots, m-1\}$, by $\mathbf{A}^m$.

Let $\mathcal{F}$ be a proper filter of the Boolean algebra of all subsets of I. Then the relation $\theta_{\mathcal{F}}$ defined on $\prod_I A_i$ by

$$\langle a, b\rangle \in \theta_{\mathcal{F}} \quad \text{iff} \quad \{i \in I \mid a(i) = b(i)\} \in \mathcal{F}$$

is an equivalence relation. We denote the block of $\theta_{\mathcal{F}}$ containing a by $a/\mathcal{F}$ and the set of blocks by $\prod_I A_i/\mathcal{F}$. The *$\mathcal{F}$-reduced product* $\prod_I \mathbf{A}_i/\mathcal{F}$ is the structure whose base set is $\prod_I A_i/\mathcal{F}$, and whose relations are defined by

$$\langle a_0/\mathcal{F}, \dots, a_{\mathrm{ar}(\mathsf{r})-1}/\mathcal{F}\rangle \in \mathsf{r}^{\prod_I \mathbf{A}_i/\mathcal{F}}$$
$$\text{iff} \quad \{i \in I \mid \langle a_0(i), \dots, a_{\mathrm{ar}(\mathsf{r})-1}(i)\rangle \in \mathsf{r}^{\mathbf{A}_i}\} \in \mathcal{F}.$$

Note that this is well defined.

Definition 2.2. Let $\mathbf{A}$ and $\mathbf{B}$ be structures. $\mathbf{B}$ is a *weak substructure* of $\mathbf{A}$ if $B \subseteq A$ and for all $\mathsf{r} \in \mathcal{R}$ it holds

$$\bar{b} \in \mathsf{r}^{\mathbf{B}} \quad \text{implies} \quad \bar{b} \in \mathsf{r}^{\mathbf{A}},$$

i.e., $\mathsf{r}^{\mathbf{B}} \subseteq \mathsf{r}^{\mathbf{A}} \cap B^{\mathrm{ar}(\mathsf{r})}$. If $\mathsf{r}^{\mathbf{B}} = \mathsf{r}^{\mathbf{A}} \cap B^{\mathrm{ar}(\mathsf{r})}$ holds for all $\mathsf{r} \in \mathcal{R}$, we call $\mathbf{B}$ a *substructure* of $\mathbf{A}$.

Definition 2.3. Let $\mathbf{A}$ and $\mathbf{B}$ be structures. A mapping $\alpha\colon \mathbf{A} \to \mathbf{B}$ is a *homomorphism* if for all $\mathsf{r} \in \mathcal{R}$ it holds

$$\bar{a} \in \mathsf{r}^{\mathbf{A}} \quad \text{implies} \quad \alpha(\bar{a}) \in \mathsf{r}^{\mathbf{B}},$$

i.e., $\alpha(\mathsf{r}^{\mathbf{A}}) \subseteq \mathsf{r}^{\mathbf{B}}$. If $\alpha(\mathsf{r}^{\mathbf{A}}) = \mathsf{r}^{\mathbf{B}}$ holds for all $\mathsf{r} \in \mathcal{R}$, we call α a *full homomorphism.* If a full homomorphism is bijective, we call it an *isomorphism* and the structures $\mathbf{A}$ and $\mathbf{B}$ are *isomorphic.*

A surjective mapping $\alpha\colon \mathbf{A} \to \mathbf{B}$ is a *strong homomorphism* if for all $\mathsf{r} \in \mathcal{R}$ it holds

$$\bar{a} \in \mathsf{r}^{\mathbf{A}} \quad \text{iff} \quad \alpha(\bar{a}) \in \mathsf{r}^{\mathbf{B}},$$

i.e., $\alpha(\mathsf{r}^{\mathbf{A}}) = \mathsf{r}^{\mathbf{B}}$ and $\alpha^{-1}(\mathsf{r}^{\mathbf{B}}) = \mathsf{r}^{\mathbf{A}}$.

A mapping $\alpha\colon \mathbf{A} \to \mathbf{B}$ is a *retraction* if it is a homomorphism and there exists a homomorphism $\alpha'\colon \mathbf{B} \to \mathbf{A}$ such that $\alpha\alpha' = \mathrm{id}_B$ holds. We call such an α' a *coretraction*, and $\mathbf{B}$ a *retract* of $\mathbf{A}$. We refer to this situation by saying "$(\alpha, \alpha')\colon \mathbf{A} \to \mathbf{B}$ is a retraction".

In Figure 2.1 a retraction between two ordered sets is shown. The right arrows depict the retraction, the left arrows depict the coretraction. We observe that the retract of a distributive lattice order can be a nonmodular lattice order.

Definition 2.4. Let I be a nonempty set with an upward directed order $\leq$, i.e., for all $i_1, i_2 \in I$ there is an $i \in I$ with $i_1, i_2 \leq i$. We call a family $(\mathbf{A}_i \mid i \in I)$ of structures indexed by I a *directed family of structures* if

$$i_1 \leq i_2 \quad \text{implies} \quad \mathbf{A}_{i_1} \text{ is a substructure of } \mathbf{A}_{i_2}.$$

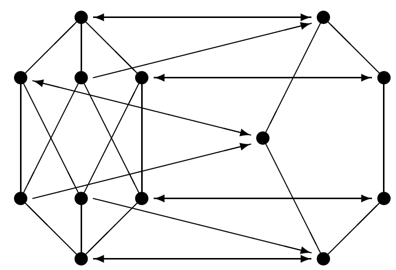

Figure 2.1: A retraction

The *direct union* $\bigcup_{(I,\leq)} \mathbf{A}_i$ of a directed family of structures is the structure whose base set is $\bigcup_I A_i$, and whose relations are defined by

$$\bar{a} \in \mathsf{r}^{\bigcup_{(I,\leq)} \mathbf{A}_i} \quad \text{iff} \quad (\exists i \in I)\ \bar{a} \in \mathsf{r}^{\mathbf{A}_i},$$

i.e., $\mathsf{r}^{\bigcup_{(I,\leq)} \mathbf{A}_i} = \bigcup_I \mathsf{r}^{\mathbf{A}_i}$.

Definition 2.5. Let I be a nonempty set with an upward directed order $\leq$, i.e., for all $i_1, i_2 \in I$ there is an $i \in I$ with $i_1, i_2 \leq i$. A *direct family of structures* is a family $(\mathbf{A}_i \mid i \in I)$ of structures indexed by I together with a family of homomorphisms

$$(\alpha_{i_1,i_2} \colon \mathbf{A}_{i_1} \to \mathbf{A}_{i_2} \mid i_1, i_2 \in I,\ i_1 \leq i_2)$$

such that $\alpha_{i,i} = \mathrm{id}_{A_i}$ for all $i \in I$ and

$$\alpha_{i_2,i_3}\alpha_{i_1,i_2} = \alpha_{i_1,i_3} \qquad \text{for all } i_1 \leq i_2 \leq i_3.$$

Let A' be the disjoint union of the base sets A_i. We introduce an equivalence relation $\sim$ on A' as follows. Given $a_1 \in A_{i_1}$ and $a_2 \in A_{i_2}$, then we set $a_1 \sim a_2$ if there is an $i \geq i_1, i_2$ such that $\alpha_{i_1,i}(a_1) = \alpha_{i_2,i}(a_2)$. Let $A := A'/{\sim}$ be the partition of A' defined by $\sim$. The *direct limit* $\lim_{(I,\leq)} \mathbf{A}_i$ of the direct family of structures is the structure whose base set is A, and whose relations are defined by

$$\langle a_0, \ldots, a_{m-1} \rangle \in \mathsf{r}^{\lim_{(I,\leq)} \mathbf{A}_i}$$

if there are representatives $a_0' \in a_0, \ldots, a_{m-1}' \in a_{m-1}$ and an $i \in I$ such that $a_0', \ldots, a_{m-1}' \in A_i$ and

$$\langle a_0', \ldots, a_{m-1}' \rangle \in \mathsf{r}^{\mathbf{A}_i}.$$

In what follows, by direct limits we mean always direct limits of direct families. The mappings $\alpha_{i^*} \colon \mathbf{A}_{i^*} \to \lim_{(I,\leq)} \mathbf{A}_i$, $i^* \in I$, are defined to map an element of A_{i^*} to its equivalence class in A and are called the *limit cone*. It is easy to see that they are homomorphisms and $\alpha_{i_2}\alpha_{i_1,i_2} = \alpha_{i_1}$ holds for all $i_1 \leq i_2$.

For a class K of structures we denote:

$\mathrm{P}K$	all products of structures from K,
$\mathrm{P}_{\mathrm{fin}}K$	all finite products of structures from K,
$\mathrm{R}K$	all retracts of structures from K,
$\mathrm{L}K$	all direct limits of structures from K,
$\mathrm{I}K$	all structures isomorphic with a structure from K.

A class K of structures is called *relational variety* if $\mathrm{R}\,\mathrm{P}\,K = K$ holds. A class K of finite structures is called *pseudovariety* if $\mathrm{R}\,\mathrm{P}_{\mathrm{fin}}\,K = K$ holds. We omit the word "relational" for pseudovarieties, since we do not encounter pseudovarieties of algebras. We remark that in the literature the concept of a variety of relational structures or of first-order structures is sometimes used in a different meaning, e.g. for classes closed under formation of homomorphic images, substructures and products. Then the concept of pseudovariety accordingly carries a different meaning.

We call a relational variety (pseudovariety, resp.) K' *finitely generated* if there is a finite set K of finite structures such that $K' = \mathrm{R}\,\mathrm{P}\,K$ ($K' = \mathrm{R}\,\mathrm{P}_{\mathrm{fin}}\,K$, resp.). The following result ensures that finitely generated pseudovarieties are exactly the finite parts of finitely generated relational varieties.

Proposition 2.6 ([Zád97]). *Let K be a finite set of finite structures, and let $\mathbf{A}$ be a finite structure. If $\mathbf{A} \in \mathrm{R}\,\mathrm{P}\,K$ then $\mathbf{A} \in \mathrm{R}\,\mathrm{P}_{\mathrm{fin}}\,K$.*

An equivalent way to define retractions is the following.

Remark 2.7. Let α be an idempotent endomorphism of $\mathbf{A}$, and let $\alpha(\mathbf{A})$ denote the substructure of $\mathbf{A}$ with base set $\alpha(A)$. Then $\alpha\colon \mathbf{A} \to \alpha(\mathbf{A})$ is a retraction with coretraction $\mathrm{id}_{\alpha(A)}$.

Vice versa, let $(\alpha, \alpha')\colon \mathbf{A} \to \mathbf{B}$ be a retraction. Then $\alpha'\alpha$ is an idempotent endomorphism of $\mathbf{A}$, and $\mathbf{B}$ is isomorphic to the substructure of $\mathbf{A}$ with base set $\alpha'(B) = (\alpha'\alpha)(A)$.

In the following two lemmas we state two basic connections between products and retracts.

Lemma 2.8. *Let $\mathbf{A}_i$, $i \in I$, be structures, and let $i^* \in I$. Then the i^*-th projection map $\pi_{i^*}\colon \prod_I \mathbf{A}_i \to \mathbf{A}_{i^*}$ is a homomorphism. If, in addition, for all $i \in I$ there is a homomorphism $\alpha_i\colon \mathbf{A}_{i^*} \to \mathbf{A}_i$, then π_{i^*} is a retraction. Especially, $\pi_{i^*}\colon \mathbf{A}^I \to \mathbf{A}$ is a retraction for any structure $\mathbf{A}$.*

Proof. It is an immediate consequence of Definitions 2.1 and 2.3 that π_{i^*} is a homomorphism. A coretraction $\alpha'\colon \mathbf{A}_{i^*} \to \prod_I \mathbf{A}_i$ is given by

$$\alpha'(a)(i) := \begin{cases} a & \text{if } i = i^*, \\ \alpha_i(a) & \text{otherwise.} \end{cases}$$

The statement about powers of structures follows from the fact that id_A is always a homomorphism. $\square$

The following can be derived straightforwardly.

Lemma 2.9. *For a class K of structures the following statements hold:*

(i) $\mathrm{I}\,K \subseteq \mathrm{R}\,K$,

(ii) $\mathrm{R}\,\mathrm{R}\,K = \mathrm{R}\,K$,

(iii) $\mathrm{I}\,\mathrm{P}\,\mathrm{P}\,K = \mathrm{I}\,\mathrm{P}\,K$ *and* $\mathrm{I}\,\mathrm{P}_{\mathrm{fin}}\,\mathrm{P}_{\mathrm{fin}}\,K = \mathrm{I}\,\mathrm{P}_{\mathrm{fin}}\,K$,

(iv) $\mathrm{P}\,\mathrm{R}\,K \subseteq \mathrm{R}\,\mathrm{P}\,K$ *and* $\mathrm{P}_{\mathrm{fin}}\,\mathrm{R}\,K \subseteq \mathrm{R}\,\mathrm{P}_{\mathrm{fin}}\,K$.

The fact that two structures are retracts of finite powers of each other is often expressed using an additional property of retractions, namely to be invertible. Here, we find it convenient to consider idempotent endomorphisms instead of retracts (cf. Remark 2.7). Lemma 2.11 and Corollary 2.12 can be formulated for retracts by obvious modifications.

Definition 2.10. Let $\alpha \in \mathrm{Func}^{(1)}(A)$ and $F \subseteq \mathrm{Func}(A)$. The map α is *invertible by F* if there exist $m \in \mathbb{N}_+$, $f_i \in F^{(1)}$ for all $i \in \{0,\dots,m-1\}$ and $f \in F^{(m)}$ such that

$$f(\alpha(f_0(a)),\dots,\alpha(f_{m-1}(a))) = a$$

holds for all $a \in A$.

In the literature, a unary term function of an algebra $\mathbb{A}$ is called invertible if it is invertible by the set of all term functions of $\mathbb{A}$ in the sense above. Here, we find it convenient to refer explicitly to F in the definition.

Lemma 2.11 (cf. [DL01, McK96]). *Let α be an idempotent endomorphism of a structure $\mathbf{A}$, and let $\mathbf{B} := \alpha(\mathbf{A})$. Then the following statements are equivalent*

(i) *α is invertible by* $\mathrm{Pol}\,\mathbf{A}$,

(ii) $\mathbf{A} \in \mathrm{R}\,\mathrm{P}_{\mathrm{fin}}\,\mathbf{B}$, *i.e., there exist $m' \in \mathbb{N}_+$ and a retraction $(\beta,\beta')\colon \mathbf{B}^{m'} \to \mathbf{A}$.*

From the proofs given in [DL01, McK96], we present here just the definitions of the required mappings, adapted to our notation. Let m, f_i and f be as in Definition 2.10. To prove "(i) implies (ii)" we set $m' := m$,

$$\begin{aligned} \beta(\langle b_0,\dots,b_{m'-1}\rangle) &:= f(b_0,\dots,b_{m'-1}), && \langle b_0,\dots,b_{m'-1}\rangle \in B^{m'},\\ \beta'(a)(i) &:= \alpha(f_i(a)), && a \in A, i \in \{0,\dots,m'-1\}. \end{aligned}$$

To prove "(ii) implies (i)" we set $m := m'$,

$$\begin{aligned} f(a_0,\dots,a_{m-1}) &:= \beta(\langle \alpha(a_0),\dots,\alpha(a_{m-1})\rangle), && a_0,\dots,a_{m-1} \in A,\\ f_i(a) &:= \beta'(a)(i), && a \in A, i \in \{0,\dots,m-1\}. \end{aligned}$$

Corollary 2.12. *Let $\mathbf{A}$ be a structure, and let $m \in \mathbb{N}_+$. Let α be an idempotent endomorphism of $\mathbf{A}^m$, and let $\mathbf{B} := \alpha(\mathbf{A}^m)$. Then the following statements are equivalent*

(i) *α is invertible by* $\mathrm{Pol}(\mathbf{A}^m)$,

(ii) $\mathbf{A} \in \mathrm{R}\,\mathrm{P}_{\mathrm{fin}}\,\mathbf{B}$.

Proof. "(i)$\Longrightarrow$(ii)". By Lemma 2.11, we have $\mathbf{A}^m \in \mathrm{R}\,\mathrm{P}_{\mathrm{fin}}\,\mathbf{B}$. By Lemma 2.8, it follows $\mathbf{A} \in \mathrm{R}\,\mathbf{A}^m$. Hence, using Lemma 2.9, $\mathbf{A} \in \mathrm{R}\,\mathrm{R}\,\mathrm{P}_{\mathrm{fin}}\,\mathbf{B} = \mathrm{R}\,\mathrm{P}_{\mathrm{fin}}\,\mathbf{B}$.

"(ii)$\Longrightarrow$(i)". Using Lemma 2.9, we have $\mathbf{A}^m \in \mathrm{P}_{\mathrm{fin}}\,\mathrm{R}\,\mathrm{P}_{\mathrm{fin}}\,\mathbf{B} = \mathrm{R}\,\mathrm{P}_{\mathrm{fin}}\,\mathbf{B}$. Hence, by Lemma 2.11, α is invertible by $\mathrm{Pol}(\mathbf{A}^m)$. □

2.2 Formulas and constructions

We analyze, for which S, the relations definable by formulas in $\Phi(S)$ behave like base relations under the constructions defined above.

Lemma 2.13. *Let $\alpha\colon \mathbf{A} \to \mathbf{B}$ be a homomorphism. Let φ be an existential-positive formula, i.e., $\varphi \in \Phi(\exists, \wedge, \vee, \approx, \mathsf{f}, \mathsf{t})$. Then $\alpha(\varphi^{\mathbf{A}}) \subseteq \varphi^{\mathbf{B}}$.*

Proof. We proceed by induction on φ. In the cases where φ is atomic, the statement of the lemma is obvious.

Case. Assume $\varphi = \varphi_1 \vee \varphi_2$. We calculate

$$\alpha(\varphi^{\mathbf{A}}) = \alpha(\varphi_1^{\mathbf{A}} \cup \varphi_2^{\mathbf{A}}) = \alpha(\varphi_1^{\mathbf{A}}) \cup \alpha(\varphi_2^{\mathbf{A}}) \subseteq \varphi_1^{\mathbf{B}} \cup \varphi_2^{\mathbf{B}} = \varphi^{\mathbf{B}}.$$

Case. Assume $\varphi = \varphi_1 \wedge \varphi_2$. We calculate

$$\alpha(\varphi^{\mathbf{A}}) = \alpha(\varphi_1^{\mathbf{A}} \cap \varphi_2^{\mathbf{A}}) \subseteq \alpha(\varphi_1^{\mathbf{A}}) \cap \alpha(\varphi_2^{\mathbf{A}}) \subseteq \varphi_1^{\mathbf{B}} \cap \varphi_2^{\mathbf{B}} = \varphi^{\mathbf{B}}.$$

Case. Assume $\varphi(\bar{x}) = (\exists x)\ \varphi'(\bar{x}, x)$. If $\bar{a} \in \varphi^{\mathbf{A}}$ then there is an $a \in A$ such that $\varphi'^{\mathbf{A}}(\bar{a}, a)$. By the induction hypothesis, $\varphi'^{\mathbf{B}}(\alpha(\bar{a}), \alpha(a))$, so $\alpha(\bar{a}) \in \varphi^{\mathbf{B}}$.

□

We may represent Lemma 2.13 as in the first line of the following table. The remaining lines summarize Lemmas 2.14–2.18.

	$\exists$	$\forall$	$\wedge$	$\vee$	$\neg$	$\approx$	f	t
homomorphisms	+		+	+		+	+	+
substructures			+	+	+	+	+	+
weak substructures	+		+	+		+	+	+
retracts	+		+	+		+	+	+
products	+		+			+	+	+
strong homomorphisms	+	+	+	+	+		+	+
limits	+		+	+		+	+	+

Lemma 2.14. (i) *Let $\mathbf{B}$ be a substructure of $\mathbf{A}$. Let φ be a quantifier-free formula, i.e., $\varphi \in \Phi(\wedge, \vee, \neg, \approx, \mathsf{f}, \mathsf{t})$. Then $\varphi^{\mathbf{B}} = \varphi^{\mathbf{A}} \cap B^{\mathrm{ar}(\varphi)}$.*

(ii) *Let $\mathbf{B}$ be a weak substructure of $\mathbf{A}$. Let φ be an existential-positive formula, i.e., $\varphi \in \Phi(\exists, \wedge, \vee, \approx, \mathsf{f}, \mathsf{t})$. Then $\varphi^{\mathbf{B}} \subseteq \varphi^{\mathbf{A}} \cap B^{\mathrm{ar}(\varphi)}$.*

Proof. (i). We proceed by induction on φ. In the cases where φ is atomic, the statement of the lemma is obvious. Let $m := \mathrm{ar}(\varphi)$.

Case. Assume $\varphi = \neg\varphi'$. We calculate

$$\varphi^{\mathbf{A}} \cap B^m = (A^m \setminus \varphi'^{\mathbf{A}}) \cap B^m = B^m \setminus (\varphi'^{\mathbf{A}} \cap B^m) = B^m \setminus \varphi'^{\mathbf{B}} = \varphi^{\mathbf{B}}.$$

Case. Assume $\varphi = \varphi_1 \wedge \varphi_2$. We calculate

$$\varphi^{\mathbf{A}} \cap B^m = \varphi_1^{\mathbf{A}} \cap \varphi_2^{\mathbf{A}} \cap B^m = \varphi_1^{\mathbf{A}} \cap B^m \cap \varphi_2^{\mathbf{A}} \cap B^m = \varphi_1^{\mathbf{B}} \cap \varphi_2^{\mathbf{B}} = \varphi^{\mathbf{B}}.$$

(ii). Since the mapping defined by $b \mapsto b$ is a homomorphism from $\mathbf{B}$ to $\mathbf{A}$, this is a consequence of Lemma 2.13. □

Lemma 2.15. *Let $(\alpha, \alpha')\colon \mathbf{A} \to \mathbf{B}$ be a retraction. Let φ be an existential-positive formula, i.e., $\varphi \in \Phi(\exists, \wedge, \vee, \approx, \mathsf{f}, \mathsf{t})$. Then $\alpha(\varphi^{\mathbf{A}}) = \varphi^{\mathbf{B}}$ and $\alpha'(\varphi^{\mathbf{B}}) \subseteq \varphi^{\mathbf{A}}$.*

Proof. Since α and α' are homomorphisms, Lemma 2.13 yields $\alpha(\varphi^{\mathbf{A}}) \subseteq \varphi^{\mathbf{B}}$ and $\alpha'(\varphi^{\mathbf{B}}) \subseteq \varphi^{\mathbf{A}}$. Now $\alpha\alpha' = \mathrm{id}_B$ implies $\alpha(\varphi^{\mathbf{A}}) = \varphi^{\mathbf{B}}$. □

Lemma 2.16. *Let $\mathbf{A} = \prod_I \mathbf{A}_i$. Let φ be a primitive-positive formula, i.e., $\varphi \in \Phi(\exists, \wedge, \approx, \mathsf{f}, \mathsf{t})$. Then*

$$\langle a_0, \ldots, a_{m-1}\rangle \in \varphi^{\mathbf{A}} \quad \textit{iff} \quad \langle a_0(i), \ldots, a_{m-1}(i)\rangle \in \varphi^{\mathbf{A}_i} \textit{ for all } i \in I.$$

Proof. We proceed by induction on φ. In the cases where φ is atomic, the statement of the lemma is obvious. Let $\bar{a} = \langle a_0, \ldots, a_{m-1}\rangle$ and

$$\bar{a(i)} = \langle a_0(i), \ldots, a_{m-1}(i)\rangle.$$

Case. Assume $\varphi = \varphi_1 \wedge \varphi_2$. It holds

$$\begin{aligned}\bar{a} \in \varphi^{\mathbf{A}} \quad \text{iff} \quad \bar{a} \in \varphi_1^{\mathbf{A}} \cap \varphi_2^{\mathbf{A}} \quad \text{iff} \quad & \bar{a(i)} \in \varphi_1^{\mathbf{A}_i} \cap \varphi_2^{\mathbf{A}_i} \text{ for all } i \in I \\ \text{iff} \quad & \bar{a(i)} \in \varphi^{\mathbf{A}_i} \text{ for all } i \in I.\end{aligned}$$

Case. Assume $\varphi(\bar{x}) = (\exists x)\ \varphi'(\bar{x}, x)$. If $\bar{a} \in \varphi^{\mathbf{A}}$ then there exists an $a \in A$ such that $\varphi'^{\mathbf{A}}(\bar{a}, a)$. Hence, for all $i \in I$ it holds $\varphi'^{\mathbf{A}_i}(\bar{a(i)}, a(i))$, so $\bar{a(i)} \in \varphi^{\mathbf{A}_i}$.

Vice versa, if $\bar{a(i)} \in \varphi^{\mathbf{A}_i}$ then there exist elements $a_i \in A_i$ such that $\varphi'^{\mathbf{A}_i}(\bar{a(i)}, a_i)$ for all $i \in I$. Define $a \in A$ by $a(i) = a_i$. Then $\varphi'^{\mathbf{A}}(\bar{a}, a)$, so $\bar{a} \in \varphi^{\mathbf{A}}$.

□

Lemma 2.17. *Let $\alpha\colon \mathbf{A} \to \mathbf{B}$ be a strong homomorphism. Let φ be a formula without $\approx$, i.e., $\varphi \in \Phi(\forall, \exists, \wedge, \vee, \neg, \mathsf{f}, \mathsf{t})$. Then for all $\bar{a} \in A^{\mathrm{ar}(\varphi)}$ it holds*

$$\bar{a} \in \varphi^{\mathbf{A}} \quad \textit{iff} \quad \alpha(\bar{a}) \in \varphi^{\mathbf{B}}.$$

Proof. We proceed by induction on φ. In the cases where φ is atomic, the statement of the lemma is obvious.

Case. Assume $\varphi = \neg\varphi'$. It holds

$$\bar{a} \in \varphi^{\mathbf{A}} \quad \text{iff} \quad \text{not } \bar{a} \in \varphi'^{\mathbf{A}} \quad \text{iff} \quad \text{not } \alpha(\bar{a}) \in \varphi'^{\mathbf{B}} \quad \text{iff} \quad \alpha(\bar{a}) \in \varphi^{\mathbf{B}}.$$

Case. Assume $\varphi = \varphi_1 \wedge \varphi_2$. It holds

$$\begin{aligned} \bar{a} \in \varphi^{\mathbf{A}} \quad &\text{iff} \quad \bar{a} \in \varphi_1^{\mathbf{A}} \text{ and } \bar{a} \in \varphi_2^{\mathbf{A}} \\ &\text{iff} \quad \alpha(\bar{a}) \in \varphi_1^{\mathbf{B}} \text{ and } \alpha(\bar{a}) \in \varphi_2^{\mathbf{B}} \quad \text{iff} \quad \alpha(\bar{a}) \in \varphi^{\mathbf{B}}. \end{aligned}$$

Case. Assume $\varphi(\bar{x}) = (\exists x)\ \varphi'(\bar{x}, x)$. If $\bar{a} \in \varphi^{\mathbf{A}}$ then there exists $a \in A$ such that ${\varphi'}^{\mathbf{A}}(\bar{a}, a)$. Hence, ${\varphi'}^{\mathbf{B}}(\alpha(\bar{a}), \alpha(a))$, so $\alpha(\bar{a}) \in \varphi^{\mathbf{B}}$.

Vice versa, if $\alpha(\bar{a}) \in \varphi^{\mathbf{B}}$ then there exists $b \in B$ such that ${\varphi'}^{\mathbf{B}}(\alpha(\bar{a}), b)$. We choose a to be any pre-image of b. Then ${\varphi'}^{\mathbf{A}}(\bar{a}, a)$, so $\bar{a} \in \varphi^{\mathbf{A}}$.

□

Lemma 2.18. *Let $(\mathbf{A}_i \mid i \in I)$ be a direct family of structures with the homomorphisms $(\alpha_{i_1,i_2} \mid i_1, i_2 \in I, i_1 \leq i_2)$. Let $\mathbf{A}$ be the direct limit $\lim_{(I,\leq)} \mathbf{A}_i$. Let φ be an existential-positive formula, i.e., $\varphi \in \Phi(\exists, \wedge, \vee, \approx, \mathsf{f}, \mathsf{t})$. Then*

$$\langle a_0, \dots, a_{m-1} \rangle \in \varphi^{\mathbf{A}}$$

if and only if there are representatives $a_0' \in a_0$, …, $a_{m-1}' \in a_{m-1}$ and an $i \in I$ such that $a_0', \dots, a_{m-1}' \in A_i$ and

$$\langle a_0', \dots, a_{m-1}' \rangle \in \varphi^{\mathbf{A}_i}.$$

Proof. We proceed by induction on φ. In the cases where φ is atomic, the statement of the lemma is obvious. Let $\bar{a} = \langle a_0, \dots, a_{m-1} \rangle$.

Case. Assume $\varphi = \varphi_1 \vee \varphi_2$. If $\bar{a} \in \varphi^{\mathbf{A}}$ we can assume $\bar{a} \in \varphi_1^{\mathbf{A}}$. Then there is a tuple $\bar{a}'$ of representatives of $\bar{a}$ and $i \in I$ such that $\bar{a}' \in \varphi_1^{\mathbf{A}_i}$, so $\bar{a}' \in \varphi^{\mathbf{A}_i}$.

Vice versa, if $\bar{a}' \in \varphi^{\mathbf{A}_i}$ we can assume $\bar{a}' \in \varphi_1^{\mathbf{A}_i}$. Then $\bar{a} \in \varphi_1^{\mathbf{A}}$, thus $\bar{a} \in \varphi^{\mathbf{A}}$.

Case. Assume $\varphi = \varphi_1 \wedge \varphi_2$. If $\bar{a} \in \varphi^{\mathbf{A}}$ then there are tuples $\bar{a}_1'$ and $\bar{a}_2'$ of representatives of $\bar{a}$ and $i_1, i_2 \in I$ such that $\bar{a}_1' \in \varphi_1^{\mathbf{A}_{i_1}}$ and $\bar{a}_2' \in \varphi_2^{\mathbf{A}_{i_2}}$. Thus, for some upper bound i of i_1 and i_2 in $(I, \leq)$ we have

$$\alpha_{i_1,i}(\bar{a}_1') = \alpha_{i_2,i}(\bar{a}_2') =: \bar{a}'.$$

By definition, $\bar{a}'$ is a tuple of representatives of $\bar{a}$ satisfying $\bar{a}' \in \varphi_1^{\mathbf{A}_i}$ and $\bar{a}' \in \varphi_2^{\mathbf{A}_i}$, so $\bar{a}' \in \varphi^{\mathbf{A}_i}$.

Vice versa, if $\bar{a}' \in \varphi^{\mathbf{A}_i}$ then $\bar{a}' \in \varphi_1^{\mathbf{A}_i}$ and $\bar{a}' \in \varphi_2^{\mathbf{A}_i}$, thus $\bar{a} \in \varphi^{\mathbf{A}}$.

Case. Assume $\varphi(\bar{x}) = (\exists x)\ \varphi'(\bar{x}, x)$. It holds

$$\bar{a} \in \varphi^{\mathbf{A}} \quad \text{iff} \quad {\varphi'}^{\mathbf{A}}(\bar{a}, a) \text{ for some } a \in A.$$

This holds if and only if there is a tuple $\bar{a}'$ of representatives of $\bar{a}$, a representative a' of a and $i \in I$ such that ${\varphi'}^{\mathbf{A}_i}(\bar{a}', a')$, and this is equivalent to $\varphi^{\mathbf{A}_i}(\bar{a}')$.

□

2.3 Axiomatizing relational varieties

In this section we introduce several types of first-order expressions: relational equations, positive equations and the more general notation $\Sigma(S)$ for equivalences of formulas of $\Phi(S)$. We characterize relational equations to be a special type of Horn-sentences and show that finitely generated pseudovarieties can be axiomatized by relational equations. This is the main result of this chapter and establishes a model-theoretic approach to relational varieties.

Definition 2.19. Let S be as in Definition 1.4. By $\Sigma(S)$ we denote the set of all first-order sentences of the form

$$(\forall \bar{x})\ \varphi_1(\bar{x}) \leftrightarrow \varphi_2(\bar{x}),$$

where $\varphi_i(\bar{x}) \in \Phi(S)$.

In particular, a *relational equation* is an equivalence of two primitive-positive formulas, i.e., an element of $\Sigma(\exists, \wedge, \approx, \mathsf{t}, \mathsf{f})$, and a *positive equation* is an equivalence of two existential-positive formulas, i.e., an element of $\Sigma(\exists, \wedge, \vee, \approx, \mathsf{t}, \mathsf{f})$.

If it causes no confusion, we write just $\varphi_1(\bar{x}) \leftrightarrow \varphi_2(\bar{x})$, or even $\varphi_1 \leftrightarrow \varphi_2$, instead of $(\forall \bar{x})\ \varphi_1(\bar{x}) \leftrightarrow \varphi_2(\bar{x})$. In Chapter 5 we give many examples of relational equations. An easy one are Equations (5.1) and (5.2) at page 49, which express the property to be an ordered set.

Remark 2.20. Assume $\wedge \in S$. Then replacing $\leftrightarrow$ by $\rightarrow$ in the foregoing definition does not change the expressive power. That is, any sentence of the form

$$\varphi_1 \rightarrow \varphi_2, \qquad \varphi_i \in \Phi(S),$$

is equivalent to an element of $\Sigma(S)$. In fact, it is equivalent to $\varphi_1 \leftrightarrow \varphi_1 \wedge \varphi_2$. Vice versa, any $\varphi_1 \leftrightarrow \varphi_2 \in \Sigma(S)$ is equivalent to $\{\varphi_1 \rightarrow \varphi_2, \varphi_2 \rightarrow \varphi_1\}$. We use this fact frequently without mentioning it explicitly, i.e., we consider an expression $\varphi_1 \rightarrow \varphi_2$ to be an element of $\Sigma(S)$.

Clearly, $\mathbf{A} \models \varphi_1 \leftrightarrow \varphi_2$ if and only if $\varphi_1^{\mathbf{A}} = \varphi_2^{\mathbf{A}}$. The analogy with algebras described in Theorem 1.6 suggests thinking about primitive-positive formulas φ as "relational terms" and the $\varphi^{\mathbf{A}}$ as "term relations". Like a usual equation states the equality of two term functions, a relational equation states the equality of two "term relations".

For a class K of structures we denote by $\mathrm{Th}_{\mathrm{re}}\, K$ ($\mathrm{Th}_{\mathrm{pe}}\, K$ resp.) the set of all relational equations (positive equations resp.) satisfied by all structures from K.

We want to compare relational equations with some known classes of first-order expressions. For clarity, we use also y to denote variables. Recall that a Horn-sentence is a first-order sentence of the form

$$(Q\bar{x})\ \bigwedge\nolimits_{j \in J} (\varphi_j \rightarrow \psi_j),$$

where $Q\bar{x}$ is any sequence of quantifiers, the φ_j are conjunctions of atomic formulas, and the ψ_j are atomic formulas. Note that some of the ψ_j can be f.

A $\forall$-$\exists$-sentence is a sentence of the form

$$(\forall \bar{x})(\exists \bar{y})\ \varphi,$$

where φ is a quantifier-free formula.

Lemma 2.21. *A first-order expression is equivalent to a relational equation if and only if it is equivalent to a $\forall$-$\exists$-Horn-sentence in which variables bounded by an $\exists$-quantifier occur only in non-negated atomic formulas, that is, to an expression of the form*

$$(\forall \bar{x})(\exists \bar{y})\ \bigwedge\nolimits_{j\in J}\Big(\varphi_j(\bar{x}) \to \psi_j(\bar{x},\bar{y})\Big), \tag{2.1}$$

where the φ_j are conjunctions of atomic formulas, and the ψ_j are atomic formulas.

Proof. "$\Longrightarrow$": Assume we are given a relational equation in the form $\varphi_1 \to \varphi_2$, with φ_i in prenex normal form:

$$(\forall \bar{x})\ \Big((\exists \bar{y}_1)\ \bigwedge\nolimits_k \psi_k(\bar{x},\bar{y}_1)\Big)\ \to\ \Big((\exists \bar{y}_2)\ \bigwedge\nolimits_j \psi_j(\bar{x},\bar{y}_2)\Big).$$

Then we transform equivalently as follows

$$(\forall \bar{x})(\forall \bar{y}_1)(\exists \bar{y}_2)\ \Big(\bigwedge\nolimits_k \psi_k(\bar{x},\bar{y}_1)\Big)\ \to\ \Big(\bigwedge\nolimits_j \psi_j(\bar{x},\bar{y}_2)\Big),$$
$$(\forall \bar{x})(\forall \bar{y}_1)(\exists \bar{y}_2)\ \bigwedge\nolimits_j\Big(\bigwedge\nolimits_k \psi_k(\bar{x},\bar{y}_1)\ \to\ \psi_j(\bar{x},\bar{y}_2)\Big).$$

Thus the relational equation is equivalent to an expression of the required form.

"$\Longleftarrow$": Assume we are given an expression of the form (2.1). We claim that it is equivalent to the set $\{\varphi_{J'} \mid J' \subseteq J\}$ of sentences with

$$\varphi_{J'} := (\forall \bar{x})(\exists \bar{y})\ \Big(\bigwedge\nolimits_{j\in J'} \varphi_j(\bar{x})\Big)\ \to\ \Big(\bigwedge\nolimits_{j\in J'} \psi_j(\bar{x},\bar{y})\Big), \qquad J' \subseteq J.$$

Clearly, (2.1) implies all $\varphi_{J'}$. To see the converse, let $\mathbf{A}$ be a structure satisfying all $\varphi_{J'}$, and let $\bar{a}$ be any tuple of elements of A of the same length as $\bar{x}$. Let J' be the set of all j such that $\mathbf{A} \models \varphi_j(\bar{a})$. Then the corresponding sentence $\varphi_{J'}$ implies $\mathbf{A} \models \bigwedge_{j\in J'} \psi_j(\bar{a},\bar{a}')$ for some tuple $\bar{a}'$. Hence,

$$\mathbf{A} \models \bigwedge\nolimits_{j\in J}\Big(\varphi_j(\bar{a}) \to \psi_j(\bar{a},\bar{a}')\Big),$$

thus $\mathbf{A}$ satisfies (2.1). Since $\varphi_{J'}$, $J' \subseteq J$, is equivalent to the relational equation

$$(\forall \bar{x})\ \Big(\bigwedge\nolimits_{j\in J'} \varphi_j(\bar{x})\Big)\ \to\ \Big((\exists \bar{y})\ \bigwedge\nolimits_{j\in J'} \psi_j(\bar{x},\bar{y})\Big),$$

the proof is complete. ☐

Well known results of model theory imply now that satisfaction of relational equations is preserved under the formation of direct unions of directed families and under the formation of reduced products.

In the following sequence of propositions we analyze for which S satisfaction of sentences from $\Sigma(S)$ is preserved under the formation of the constructions defined above.

Proposition 2.22. *Let* $\mathbf{B}$ *be a substructure of* $\mathbf{A}$*, and let*

$$\varphi_1 \leftrightarrow \varphi_2 \in \Sigma(\wedge, \vee, \neg, \approx, \mathsf{f}, \mathsf{t}).$$

If $\mathbf{A} \models \varphi_1 \leftrightarrow \varphi_2$ *then* $\mathbf{B} \models \varphi_1 \leftrightarrow \varphi_2$.

Proof. Let $m := \mathrm{ar}(\varphi_1) = \mathrm{ar}(\varphi_2)$. Using Lemma 2.14(i), we obtain $\varphi_1^{\mathbf{B}} = \varphi_1^{\mathbf{A}} \cap B^m = \varphi_2^{\mathbf{A}} \cap B^m = \varphi_2^{\mathbf{B}}$. □

Proposition 2.23. *Let* $(\alpha, \alpha')\colon \mathbf{A} \to \mathbf{B}$ *be a retraction.*

(i) *Let* $\varphi_1 \leftrightarrow \varphi_2 \in \Sigma(\exists, \wedge, \vee, \approx, \mathsf{f}, \mathsf{t})$*. If* $\mathbf{A} \models \varphi_1 \leftrightarrow \varphi_2$ *then* $\mathbf{B} \models \varphi_1 \leftrightarrow \varphi_2$.

(ii) *Let* $\varphi_1 \leftrightarrow \varphi_2 \in \Sigma(\wedge, \vee, \neg, \approx, \mathsf{f}, \mathsf{t})$*. If* $\mathbf{A} \models \varphi_1 \leftrightarrow \varphi_2$ *then* $\mathbf{B} \models \varphi_1 \leftrightarrow \varphi_2$.

Proof. (i). Using Lemma 2.15, we obtain $\varphi_1^{\mathbf{B}} = \alpha(\varphi_1^{\mathbf{A}}) = \alpha(\varphi_2^{\mathbf{A}}) = \varphi_2^{\mathbf{B}}$.

(ii). Immediate by Remark 2.7 and Proposition 2.22. □

Proposition 2.24. *Let* $\mathbf{A} = \prod_I \mathbf{A}_i$.

(i) *Let* $\varphi_1 \leftrightarrow \varphi_2 \in \Sigma(\exists, \wedge, \approx, \mathsf{f}, \mathsf{t})$*. If* $\mathbf{A}_i \models \varphi_1 \leftrightarrow \varphi_2$ *for all* $i \in I$*, then* $\mathbf{A} \models \varphi_1 \leftrightarrow \varphi_2$.

Let, in addition, $i^* \in I$ *be such that there is a homomorphism* $\alpha_i\colon \mathbf{A}_{i^*} \to \mathbf{A}_i$ *for all* $i \in I$.

(ii) *Let* $\varphi_1 \leftrightarrow \varphi_2 \in \Sigma(\exists, \wedge, \vee, \approx, \mathsf{f}, \mathsf{t})$*. If* $\mathbf{A} \models \varphi_1 \leftrightarrow \varphi_2$ *then* $\mathbf{A}_{i^*} \models \varphi_1 \leftrightarrow \varphi_2$.

(iii) *Let* $\varphi_1 \leftrightarrow \varphi_2 \in \Sigma(\wedge, \vee, \neg, \approx, \mathsf{f}, \mathsf{t})$*. If* $\mathbf{A} \models \varphi_1 \leftrightarrow \varphi_2$ *then* $\mathbf{A}_{i^*} \models \varphi_1 \leftrightarrow \varphi_2$.

Proof. (i). Let $m := \mathrm{ar}(\varphi_1) = \mathrm{ar}(\varphi_2)$. Using Lemma 2.16 we obtain

$$\begin{aligned} \langle a_0, \ldots, a_{m-1} \rangle \in {\varphi_1}^{\mathbf{A}} \quad & \text{iff} \quad \langle a_0(i), \ldots, a_{m-1}(i) \rangle \in \varphi_1^{\mathbf{A}_i} \text{ for all } i \in I \\ & \text{iff} \quad \langle a_0(i), \ldots, a_{m-1}(i) \rangle \in \varphi_2^{\mathbf{A}_i} \text{ for all } i \in I \\ & \text{iff} \quad \langle a_0, \ldots, a_{m-1} \rangle \in \varphi_2^{\mathbf{A}}. \end{aligned}$$

(ii),(iii). Immediate by Lemma 2.8 and Proposition 2.23. □

Proposition 2.25. *Let* $\alpha\colon \mathbf{A} \to \mathbf{B}$ *be a strong homomorphism.*

(i) *Let* $\varphi_1 \leftrightarrow \varphi_2 \in \Sigma(\forall, \exists, \wedge, \vee, \neg, \mathsf{f}, \mathsf{t})$*. It holds* $\mathbf{A} \models \varphi_1 \leftrightarrow \varphi_2$ *if and only if* $\mathbf{B} \models \varphi_1 \leftrightarrow \varphi_2$.

(ii) *Let* $\varphi_1 \leftrightarrow \varphi_2 \in \Sigma(\exists, \wedge, \vee, \approx, \mathsf{f}, \mathsf{t})$*. If* $\mathbf{A} \models \varphi_1 \leftrightarrow \varphi_2$ *then* $\mathbf{B} \models \varphi_1 \leftrightarrow \varphi_2$.

(iii) *Let* $\varphi_1 \leftrightarrow \varphi_2 \in \Sigma(\wedge, \vee, \neg, \approx, \mathsf{f}, \mathsf{t})$*. If* $\mathbf{A} \models \varphi_1 \leftrightarrow \varphi_2$ *then* $\mathbf{B} \models \varphi_1 \leftrightarrow \varphi_2$.

Proof. (i). We use Lemma 2.17.
"$\Longrightarrow$": $\varphi_1^{\mathbf{B}} = \alpha(\varphi_1^{\mathbf{A}}) = \alpha(\varphi_2^{\mathbf{A}}) = \varphi_2^{\mathbf{B}}$.
"$\Longleftarrow$": $\varphi_1^{\mathbf{A}} = \alpha^{-1}(\varphi_1^{\mathbf{B}}) = \alpha^{-1}(\varphi_2^{\mathbf{B}}) = \varphi_2^{\mathbf{A}}$.

(ii),(iii). Since any strong homomorphism is a retraction, this is immediate by Proposition 2.23. □

Proposition 2.26. *Let* $(\mathbf{A}_i \mid i \in I)$ *be a direct family of structures, and* $\mathbf{A} = \lim_{(I,\leq)} \mathbf{A}_i$. *Let* $\varphi_1 \leftrightarrow \varphi_2 \in \Sigma(\exists, \wedge, \vee, \approx, \mathsf{f}, \mathsf{t})$. *If* $\mathbf{A}_i \models \varphi_1 \leftrightarrow \varphi_2$ *for all* $i \in I$, *then* $\mathbf{A} \models \varphi_1 \leftrightarrow \varphi_2$.

Proof. We use Lemma 2.18. Let $\langle a_0, \dots, a_{m-1}\rangle \in \varphi_1^{\mathbf{A}}$. Then there are representatives $a_0' \in a_0, \dots, a_{m-1}' \in a_{m-1}$ and an $i \in I$ such that $a_0', \dots, a_{m-1}' \in A_i$ and $\langle a_0', \dots, a_{m-1}'\rangle \in \varphi_1^{\mathbf{A}_i}$. Hence, $\langle a_0', \dots, a_{m-1}'\rangle \in \varphi_2^{\mathbf{A}_i}$, so $\langle a_0, \dots, a_{m-1}\rangle \in \varphi_2^{\mathbf{A}}$. By symmetry, $\varphi_1^{\mathbf{A}} = \varphi_2^{\mathbf{A}}$. □

We remark that it is not possible to "combine" items in the foregoing lemmas. E.g. Proposition 2.23(i),(ii) does not imply that satisfaction of sentences from $\Sigma(\exists, \wedge, \vee, \neg, \approx, \mathsf{f}, \mathsf{t})$ is preserved under retractions. To see that the foregoing lemmas cover all classes of expressions definable in the form $\Sigma(S')$ is not difficult. But, one has to provide counterexamples for all S'' such that preservation of $\Sigma(S')$ is not stated for any $S' \supseteq S''$. In many such cases, we can apply the fact that the property of a structure to have at least a prescribed finite number of elements can be described by an expression in $\Sigma(\exists, \wedge, \neg, \approx)$, but this property is not preserved under the construction in question.

Propositions 2.23(i) and 2.24(i) yield that for a class K of structures it holds $\mathrm{R}\,\mathrm{P}\,K \subseteq \mathrm{Mod}\,\mathrm{Th}_{\mathrm{re}}\,K$. Our next two theorems state that, under certain finiteness conditions, equality holds.

Theorem 2.27. *Let* $\mathcal{R}$ *be a finite type, let* $\mathbf{A}$ *be a finite structure, and let* K *be a class of structures. If* $\mathbf{A} \in \mathrm{Mod}\,\mathrm{Th}_{\mathrm{re}}\,K$ *then* $\mathbf{A} \in \mathrm{R}\,\mathrm{P}\,K$.

Assuming K to be a finite class of finite structures, we can drop the finiteness condition on $\mathcal{R}$.

Theorem 2.28. *Let* $\mathbf{A}$ *be a finite structure, and let* K *be a finite class of finite structures. If* $\mathbf{A} \in \mathrm{Mod}\,\mathrm{Th}_{\mathrm{re}}\,K$ *then* $\mathbf{A} \in \mathrm{R}\,\mathrm{P}_{\mathrm{fin}}\,K$.

For the proofs we need the following notations. Let $\mathbf{A}$ be a finite structure, and let $\bar{a} = \langle a_0, \dots, a_{m-1}\rangle$ be a tuple containing each element of A exactly once, and let $\bar{x} = \langle x_0, \dots, x_{m-1}\rangle$. The *diagram* $\Phi_{\mathrm{d}}(\mathbf{A}, \bar{a})$ is defined to be the following set of atomic formulas

$$\{\psi(\bar{x}) = \mathsf{r}(x_{j_1}, \dots, x_{j_{\mathrm{ar}(\mathsf{r})}}) \mid \mathsf{r} \in \mathcal{R},\ (a_{j_1}, \dots, a_{j_{\mathrm{ar}(\mathsf{r})}}) \in \mathsf{r}^{\mathbf{A}}\}.$$

If $\Phi_{\mathrm{d}}(\mathbf{A}, \bar{a})$ is finite, the *diagram formula* $\varphi_{\mathbf{A},\bar{a}}$ is defined by

$$\varphi_{\mathbf{A},\bar{a}}(\bar{x}) := \bigwedge_{\psi \in \Phi_{\mathrm{d}}(\mathbf{A},\bar{a})} \psi(\bar{x}).$$

If $\mathbf{B}$ is a structure, $\bar{b} := \langle b_0, \dots, b_{m-1}\rangle \in B^m$, then the mapping $\alpha\colon \mathbf{A} \to \mathbf{B}$ defined by $\alpha(a_j) := b_j$, $j \in \{0, \dots, m-1\}$, is a homomorphism if and only if $\mathbf{B} \models \psi(\bar{b})$ holds for all $\psi \in \Phi_{\mathrm{d}}(\mathbf{A}, \bar{a})$ or, equivalently,

$$\mathbf{B} \models \varphi_{\mathbf{A},\bar{a}}(\bar{b})$$

provided the diagram formula exists. In particular, $\mathbf{A} \models \varphi_{\mathbf{A},\bar{a}}(\bar{a})$.

In what follows, we employ a classical compactness argument involving the concept of the inverse limit of an inverse family of sets (see, e.g., [Grä79, §21]).

Definition 2.29. Let I be a nonempty set with an upward directed order $\leq$, i.e., for all $i_1, i_2 \in I$ there is an $i \in I$ with $i_1, i_2 \leq i$. An *inverse family of sets* is a family $(A_i \mid i \in I)$ of sets together with mappings $\lambda_{i_1,i_2} \colon A_{i_1} \to A_{i_2}$ for all $i_1 \geq i_2$, such that $\lambda_{i,i} = \mathrm{id}_{A_i}$ holds for all $i \in I$ and

$$\lambda_{i_2,i_3}\lambda_{i_1,i_2} = \lambda_{i_1,i_3} \qquad \text{for all } i_1 \geq i_2 \geq i_3.$$

The *inverse limit* of the inverse family of sets is the set of all elements a of $\prod_I A_i$ satisfying

$$\lambda_{i_1,i_2}(a(i_1)) = a(i_2) \qquad \text{for all } i_1 \geq i_2.$$

Theorem 2.30 ([Grä79]). *The inverse limit of an inverse family of finite, nonempty sets is nonempty.*

Corollary 2.31 (cf. [Grä79, Thm. 21.6]). *Let* $\mathbf{A}$ *be a finite structure, and let* $\mathbf{B}$ *be a structure. If for all finite weak substructures* $\mathbf{C}$ *of* $\mathbf{B}$ *there is a homomorphism from* $\mathbf{C}$ *into* $\mathbf{A}$*, then there is a homomorphism from* $\mathbf{B}$ *into* $\mathbf{A}$*.*

Because we use many slight modifications of the foregoing corollary, which are obtained by obvious modifications of its proof, we present the proof (cf. [Grä79]).

Proof. Let $\mathcal{C}$ be the set of all finite weak substructures $\mathbf{C}$ of $\mathbf{B}$ ordered by $\subseteq$, where $\mathbf{C}_2 \subseteq \mathbf{C}_1$ denotes that $\mathbf{C}_2$ is a weak substructure of $\mathbf{C}_1$. For $\mathbf{C} \in \mathcal{C}$ let $H_{\mathbf{C}}$ be the set of all homomorphisms from $\mathbf{C}$ into $\mathbf{A}$. For $\mathbf{C}_1, \mathbf{C}_2 \in \mathcal{C}$ with $\mathbf{C}_2 \subseteq \mathbf{C}_1$ define $\lambda_{\mathbf{C}_1,\mathbf{C}_2} \colon H_{\mathbf{C}_1} \to H_{\mathbf{C}_2}$ by setting $\lambda_{\mathbf{C}_1,\mathbf{C}_2}(\alpha)$ to be the restriction of α to C_2. The sets $(H_{\mathbf{C}} \mid \mathbf{C} \in \mathcal{C})$ and the mappings $\lambda_{\mathbf{C}_1,\mathbf{C}_2}$, $\mathbf{C}_1, \mathbf{C}_2 \in \mathcal{C}$, $\mathbf{C}_2 \subseteq \mathbf{C}_1$, establish an inverse family of finite nonempty sets. By Theorem 2.30 there is an element $\bar{\alpha}$ in the inverse limit. Define $\alpha^* \colon \mathbf{B} \to \mathbf{A}$ by

$$\alpha^*(b) := (\bar{\alpha}(\mathbf{C}))(b), \qquad \text{for any } \mathbf{C} \in \mathcal{C} \text{ with } b \in C.$$

It is easy to check that α^* is well defined, i.e., it is independent of the choice of $\mathbf{C}$, and that it is a homomorphism. $\square$

Proof of Theorem 2.27. We can assume that K is a set of structures. Indeed, if K is a proper class, we choose for each relational equation φ with $K \not\models \varphi$ one structure $\mathbf{B} \in K$ such that $\mathbf{B} \not\models \varphi$. Since all relational equations form a set, this gives a set K' of structures satisfying $\mathrm{Th}_{\mathrm{re}}\, K = \mathrm{Th}_{\mathrm{re}}\, K'$. It follows $\mathbf{A} \in \mathrm{R\,P}\, K' \subseteq \mathrm{R\,P}\, K$, once the theorem is proved for K'.

First, we construct a product $\mathbf{B}^*$ of structures from K, and a homomorphism $\alpha' \colon \mathbf{A} \to \mathbf{B}^*$ which serves as a coretraction. We set

$$I := \{(\mathbf{B}, \alpha) \mid \mathbf{B} \in K,\ \alpha \colon \mathbf{A} \to \mathbf{B} \text{ is a homomorphism}\}.$$

For $i \in I$ we refer to the components of i by $\mathbf{B}_i$ and α_i. We denote $\mathbf{B}^* := \prod_I \mathbf{B}_i$ and define $\alpha' \colon \mathbf{A} \to \mathbf{B}^*$ by

$$\alpha'(a)(i) := \alpha_i(a), \qquad a \in A, i \in I.$$

Clearly, α' is a homomorphism.

It remains to construct a homomorphism $\alpha\colon \mathbf{B}^* \to \mathbf{A}$ satisfying $\alpha\alpha' = \mathrm{id}_A$. Let $\bar{a} = \langle a_0, \ldots, a_{m-1}\rangle$ be a tuple containing each element of A exactly once, and let $\bar{x} = \langle x_0, \ldots, x_{m-1}\rangle$.

Claim (*). For all primitive-positive formulas $\varphi(\bar{x})$ with $\mathbf{B}^* \models \varphi(\alpha'(\bar{a}))$ we have $\mathbf{A} \models \varphi(\bar{a})$.

Proof of Claim (*). Recall the notion of the diagram formula $\varphi_{\mathbf{A},\bar{a}}$. If $\mathbf{B} \in K$ and $\bar{b} \in B^m$ with $\mathbf{B} \models \varphi_{\mathbf{A},\bar{a}}(\bar{b})$, then $\bar{b} = \alpha_i(\bar{a})$ for some $i \in I$. This implies, by Lemma 2.16 and $\mathbf{B}^* \models \varphi(\alpha'(\bar{a}))$, that $\mathbf{B} \models \varphi(\bar{b})$. Hence,

$$\varphi_{\mathbf{A},\bar{a}} \to \varphi \in \mathrm{Th}_{\mathrm{re}}\, K.$$

Now the assumption of the theorem and $\mathbf{A} \models \varphi_{\mathbf{A},\bar{a}}(\bar{a})$ imply $\mathbf{A} \models \varphi(\bar{a})$. This completes the proof of (*).

Setting $\varphi = \mathsf{f}$ it follows that I is nonempty. Let $a_j \neq a_k$. Using $\varphi = (x_j \approx x_k)$, (*) yields $\alpha'(a_j) \neq \alpha'(a_k)$, so α' is injective.

Claim (**). For all finite weak substructures $\mathbf{C}$ of $\mathbf{B}^*$ with $\alpha'(A) \subseteq C$ there is a homomorphism $\alpha_{\mathbf{C}}\colon \mathbf{C} \to \mathbf{A}$ such that $\alpha_{\mathbf{C}}\alpha' = \mathrm{id}_A$.

Proof of Claim (**). Let $\bar{c} = \langle c_0, \ldots, c_{l-1}\rangle$ be a tuple containing each element of C exactly once. Since α' is injective, we can choose $\bar{c}$ such that $\alpha'(\bar{a}) = \langle c_0, \ldots, c_{m-1}\rangle$. We define

$$\varphi(x_0, \ldots, x_{m-1}) := (\exists x_m, \ldots, x_{l-1})\ \varphi_{\mathbf{C},\bar{c}}(x_0, \ldots, x_{l-1}).$$

Since, $\mathbf{C} \models \varphi_{\mathbf{C},\bar{c}}(\bar{c})$, it holds $\mathbf{C} \models \varphi(c_0, \ldots, c_{m-1})$. Lemma 2.14(ii) and $\alpha'(\bar{a}) = \langle c_0, \ldots, c_{m-1}\rangle$ yield

$$\mathbf{B}^* \models \varphi(\alpha'(\bar{a})).$$

By (*), we obtain $\mathbf{A} \models \varphi(\bar{a})$, i.e., there exist $a_m, \ldots, a_{l-1} \in A$ such that

$$\mathbf{A} \models \varphi_{\mathbf{C},\bar{c}}(a_0, \ldots, a_{l-1}).$$

Now $\alpha_{\mathbf{C}}$ defined by $\alpha_{\mathbf{C}}(c_j) = a_j$, $j \in \{0, \ldots, l-1\}$, satisfies Claim (**).

The proof of Corollary 2.31 remains valid, when we set $\mathcal{C}$ to be only weak substructures satisfying $\alpha'(A) \subseteq C$ instead of all weak substructures $\mathbf{C}$ of $\mathbf{B}^*$, and set $H_{\mathbf{C}}$ to be only homomorphisms satisfying $\alpha_{\mathbf{C}}\alpha' = \mathrm{id}_A$ instead of all homomorphisms. Hence, there is a homomorphism $\alpha\colon \mathbf{B}^* \to \mathbf{A}$ satisfying $\alpha\alpha' = \mathrm{id}_A$. Now, α is a retraction with coretraction α', hence, $\mathbf{A} \in \mathrm{R\,P}\, K$. □

Proof of Theorem 2.28. We modify the proof of Theorem 2.27. Define I, $\mathbf{B}^*$ and α' as above. Obviously, now I and $\mathbf{B}^*$ are finite.

Proof of Claim (*). The proof of Theorem 2.27 does not work here, since now in general the diagram $\Phi_{\mathrm{d}}(\mathbf{A}, \bar{a})$ is infinite and the diagram formula $\varphi_{\mathbf{A},\bar{a}}$ is not defined. Let

$$J := \{(\mathbf{B}, \bar{b}) \mid \mathbf{B} \in K,\ \bar{b} \in B^m,\ \mathbf{B} \not\models \varphi(\bar{b})\}.$$

Obviously, J is finite. Let $j = (\mathbf{B}, \bar{b}) \in J$ and $\alpha_j\colon \mathbf{A} \to \mathbf{B}$ be defined by $\alpha_j(\bar{a}) = \bar{b}$. By Lemma 2.16 and $\mathbf{B}^* \models \varphi(\alpha'(\bar{a}))$, we have that for all $i \in I$ it holds

$\mathbf{B}_i \models \varphi(\alpha_i(\bar{a}))$. Hence, α_j is no homomorphism, i.e., there is a $\psi_j(\bar{x}) \in \Phi_\mathrm{d}(\mathbf{A}, \bar{a})$ with $\mathbf{B} \not\models \psi_j(\bar{b})$. We define a "reduced diagram formula" by

$$\varphi'_{\mathbf{A},\bar{a}}(\bar{x}) := \bigwedge_{j \in J} \psi_j(\bar{x}).$$

Now $K \models \varphi'_{\mathbf{A},\bar{a}} \to \varphi$ and $\mathbf{A} \models \varphi'_{\mathbf{A},\bar{a}}(\bar{a})$. The assumption of the theorem yields $\mathbf{A} \models \varphi(\bar{a})$. This completes the proof of (*).

The rest of the proof remains valid with the one minor change that we have to restrict the structures $\mathbf{C}$ occurring in Claim (**) to such finite weak substructures of $\mathbf{B}^*$ where the diagram $\Phi_\mathrm{d}(\mathbf{C}, \bar{c})$ is finite. □

An example, that we cannot drop the condition "$\mathbf{A}$ finite" in Theorems 2.27 and 2.28 is given by Example 5.7. The following two examples show that not both K and $\mathcal{R}$ can be infinite. Actually, the examples are valid for all first-order sentences instead of relational equations.

Example 2.32. Let $\mathcal{R} = \{\mathsf{r}_j \mid j \in \mathbb{N}_+\}$ and $\mathrm{ar}(\mathsf{r}_j) = 1$. Let $A = \{a\}$ and define $\mathbf{A}$ by ${\mathsf{r}_j}^{\mathbf{A}} = \{\langle a \rangle\}$, $j \in \mathbb{N}_+$. Let K consist of one structure $\mathbf{B}$ defined by $B = \{b_i \mid i \in \mathbb{N}_+\}$ and ${\mathsf{r}_j}^{\mathbf{B}} = \{\langle b_i \rangle \mid i \neq j\}$.

Since there is no homomorphism of $\mathbf{A}$ into $\mathbf{B}$, there is no homomorphism of $\mathbf{A}$ into a power of $\mathbf{B}$, hence $\mathbf{A} \notin \mathrm{R}\,\mathrm{P}\,K$.

Let $\varphi_1 \leftrightarrow \varphi_2$ be any relational equation satisfied by $\mathbf{B}$. Let $\mathcal{R}'$ be the finite set of relation symbols occurring in $\varphi_1 \leftrightarrow \varphi_2$, and let $\mathbf{A}'$ ($\mathbf{B}'$ resp.) be the $\mathcal{R}'$-reduct of $\mathbf{A}$ ($\mathbf{B}$ resp.). We define $\alpha\colon \mathbf{B}' \to \mathbf{A}'$ by $\alpha(b) = a$, $b \in B$, and $\alpha'\colon \mathbf{A}' \to \mathbf{B}'$ by $\alpha(a) = b_{i^*}$, where i^* is any integer such that $\mathsf{r}_{i^*} \notin \mathcal{R}'$. Obviously, $(\alpha, \alpha')\colon \mathbf{B}' \to \mathbf{A}'$ is a retraction. We conclude that $\mathbf{B}$ and, in turn, $\mathbf{B}'$, $\mathbf{A}'$ and finally $\mathbf{A}$ satisfy $\varphi_1 \leftrightarrow \varphi_2$, i.e., $\mathbf{A} \in \mathrm{Mod}\,\mathrm{Th}_\mathrm{re}\,K$.

Example 2.33. Let $\mathcal{R}$ and $\mathbf{A}$ as in Example 2.32 above. Let K consist of structures $\mathbf{B}_1, \mathbf{B}_2, \ldots$ defined by $B_i = \{b_i\}$ and

$${\mathsf{r}_j}^{\mathbf{B}_i} = \begin{cases} \{\langle b_i \rangle\} & j \leq i \\ \emptyset & j > i \end{cases}, \qquad i, j \in \mathbb{N}_+.$$

Since there is no homomorphism of $\mathbf{A}$ into any $\mathbf{B}_i$, there is no homomorphism of $\mathbf{A}$ into a structure from $\mathrm{P}\,K$, hence $\mathbf{A} \notin \mathrm{R}\,\mathrm{P}\,K$.

Let $\varphi_1 \leftrightarrow \varphi_2$ be any relational equation satisfied by K. Let $\mathcal{R}'$ and $\mathbf{A}'$ be as in Example 2.32 above. Let i^* be such that $\mathcal{R}' \subseteq \{\mathsf{r}_1, \ldots, \mathsf{r}_{i^*}\}$, and let $\mathbf{B}'_{i^*}$ be the $\mathcal{R}'$-reduct of $\mathbf{B}_{i^*}$. Then $\mathbf{A}'$ is isomorphic to $\mathbf{B}'_{i^*}$, and, as in the example above, we conclude $\mathbf{A} \in \mathrm{Mod}\,\mathrm{Th}_\mathrm{re}\,K$.

Without giving formal definitions, we discuss the question, whether we can drop finiteness conditions from Theorems 2.27 and 2.28 if we allow infinite formulas.

We extend the notion of a primitive-positive formula by allowing infinite conjunctions. Let now a relational equation be the equivalence statement of two such primitive-positive formulas, then we achieve the conclusion of Theorems 2.27 and 2.28 under the premise

> Let $\mathbf{A}$ be a finite structure, and let K be a class of structures.

The proof is as for Theorem 2.27 with obvious modifications. If we allow, in addition, infinitely many $\exists$-quantifiers and infinitely many free variables in primitive-positive formulas, we can drop the finiteness condition on $\mathbf{A}$.

Let κ be an infinite cardinal. If we allow in primitive-positive formulas only conjunctions of at most κ many atomic formulas then we achieve the conclusion of Theorems 2.27 and 2.28 under the premises

> Let $\mathcal{R}$ be a type of at most κ many symbols. Let $\mathbf{A}$ be a finite structure, and let K be a class of structures.

or

> Let $\mathbf{A}$ be a finite structure, and let K be a class of at most κ many structures, each of size at most κ.

Again, the proof is as for Theorems 2.27 and 2.28 with obvious modifications. Variants of the Examples 2.32 and 2.33 show that we can not go further.

The results of this section have parallels with the theory of quasivarieties. A universal Horn-sentence is a first-order sentence of the form

$$(\forall \bar{x}) \ \bigwedge_{j \in J} (\varphi_j \to \psi_j), \tag{2.2}$$

where the φ_j are conjunctions of atomic formulas, and the ψ_j are atomic formulas. A class of structures is axiomatizable by universal Horn-sentences if and only if it is closed under formation of reduced products and substructures, or, equivalently, if and only if it is closed under formation of direct limits of direct families, substructures and products.

Let a finitary implication be a universal Horn-sentence but allowing infinite conjunctions, and let an implication be a universal Horn-sentence but allowing infinite conjunctions and infinitely many variables. A class of structures is axiomatizable by finitary implications if and only if it is closed under formation of unions of directed families, substructures and products. A class of structures is axiomatizable by implications if and only if it is closed under formation of substructures and products, such classes are called quasivarieties. These results can be found for classes of algebras in [Wec92] and carry over to arbitrary first-order structures, see [BS81] (universal Horn-sentences) and [Hod97] (implications).

In the Theorems 2.27 and 2.28, we made several finiteness conditions on the structures involved. I expect that a class of structures is axiomatizable by relational equations if and only if it is closed under formation of direct limits of direct families, retracts and products. Having in mind that for any class K of structures it holds $\mathrm{L}\,\mathrm{R}\,K = \mathrm{L}\,K$, we state the following problem.

Problem 2.34. Let $\mathbf{A}$ be a structure and let K be a class of structures. Does $\mathbf{A} \in \mathrm{Mod}\,\mathrm{Th}_{\mathrm{re}}\,K$ implies $\mathbf{A} \in \mathrm{L}\,\mathrm{P}\,K$?

We close this section with a partial result in this direction.

Theorem 2.35. *Let $\mathcal{R}$ be a finite type, let $\mathbf{A}$ be a structure, and let K be a finite class of finite structures. If $\mathbf{A} \in \mathrm{Mod}\,\mathrm{Th}_{\mathrm{re}}\,K$ then $\mathbf{A} \in \mathrm{L}\,\mathrm{P}\,K$.*

Proof. We construct a direct family of structures from $\mathrm{P}\,K$ indexed by finite subsets of A. Let $\mathcal{A}$ be the set of all finite, nonempty subsets of A, ordered by inclusion.

Let $A' \in \mathcal{A}$ and let $\mathbf{A}'$ be the substructure of $\mathbf{A}$ with base set A'. We set

$$I_{A'} := \{(\mathbf{B}, \alpha) \mid \mathbf{B} \in K,\ \alpha\colon \mathbf{A}' \to \mathbf{B} \text{ is a homomorphism}\}.$$

For $i \in I_{A'}$, we refer to the components of i by $\mathbf{B}_i$ and α_i. We denote

$$\mathbf{B}_{A'} := \prod\nolimits_{I_{A'}} \mathbf{B}_i,$$

and define $\alpha'_{A'}\colon \mathbf{A}' \to \mathbf{B}_{A'}$ by

$$\alpha'_{A'}(a)(i) := \alpha_i(a), \qquad a \in A', i \in I_{A'}.$$

Clearly, $\alpha'_{A'}$ is a homomorphism and, since K is a finite class of finite structures, $\mathbf{B}_{A'}$ is finite.

Claim. There is a homomorphism $\alpha_{A'}\colon \mathbf{B}_{A'} \to \mathbf{A}$ such that $\alpha_{A'}\alpha'_{A'} = \mathrm{id}_{A'}$.

Proof of the claim. Let $\bar{a}$ be a tuple containing each element of A' exactly once. As in the proof of Claim (*) in Theorem 2.27, we obtain that the diagram formula $\varphi_{\mathbf{A}',\bar{a}}$ exists, and conclude that $I_{A'}$ is nonempty and that for any primitive-positive formula φ we have $\mathbf{B}_{A'} \models \varphi(\alpha'_{A'}(\bar{a}))$ implies $\mathbf{A} \models \varphi(\bar{a})$. Since $\mathbf{B}_{A'}$ is finite, the arguments of the proof of Claim (**) in Theorem 2.27 complete the proof of the Claim.

Let $A' \in \mathcal{A}$. We set $\overline{A'} := \alpha_{A'}(B_{A'})$. By construction, $A' \subseteq \overline{A'}$ and, since $B_{A'}$ is finite, $\overline{A'}$ is finite. We define recursively

$$\tilde{A}' := A' \cup \bigcup_{\substack{A'' \in \mathcal{A} \\ A'' \subset A'}} \overline{\tilde{A}''}.$$

By construction, $\tilde{A}'$ is finite.

Now we are ready to set up the direct family of structures. For $A' \in \mathcal{A}$ we set $\mathbf{C}_{A'} := \mathbf{B}_{\tilde{A}'}$ and $\beta_{A'} := \alpha_{\tilde{A}'}$ and $\beta'_{A'} := \alpha'_{\tilde{A}'}$. We consider the family of structures $(\mathbf{C}_{A'} \mid A' \in \mathcal{A})$ together with the mappings

$$\gamma_{A_1,A_2}\colon \mathbf{C}_{A_1} \to \mathbf{C}_{A_2}, \qquad \text{for all } A_1, A_2 \in \mathcal{A} \text{ with } A_1 \subseteq A_2,$$

defined by $\gamma_{A_1,A_2} := \mathrm{id}_{C_{A_1}}$, for $A_1 = A_2$, and

$$\gamma_{A_1,A_2} := \beta'_{A_2}\beta_{A_1}, \qquad \text{for } A_1 \subset A_2.$$

Note that this is defined since, by construction, $\beta_{A_1}(C_{A_1}) = \overline{\tilde{A}_1} \subseteq \tilde{A}_2$. We may depict the situation as in the following commutative diagram.

$$\begin{tikzcd}
\mathbf{C}_{A_1} \arrow[r, "\gamma_{A_1,A_2}"] \arrow[dr, "\beta_{A_1}"] & \mathbf{C}_{A_2} \\
\tilde{A}_1 \arrow[u, "\beta'_{A_1}"] \arrow[r, "\subseteq"'] & \tilde{A}_2 \arrow[u, "\beta'_{A_2}"']
\end{tikzcd}$$

Since γ_{A_1,A_2} is a composition of homomorphisms, it is a homomorphism. If $A_1, A_2, A_3 \in \mathcal{A}$ are such that $A_1 \subseteq A_2 \subseteq A_3$, then $\gamma_{A_2,A_3}\gamma_{A_1,A_2} = \gamma_{A_1,A_3}$. Indeed, this is trivial if $A_1 = A_2$ or $A_2 = A_3$ and otherwise we have

$$\gamma_{A_2,A_3}\gamma_{A_1,A_2} = \beta'_{A_3}\beta_{A_2}\beta'_{A_2}\beta_{A_1} = \beta'_{A_3}\beta_{A_1} = \gamma_{A_1,A_3}.$$

Hence, we have a direct family of structures. We claim that its direct limit is isomorphic to $\mathbf{A}$.

Let $\mathbf{C} := \lim_{\mathcal{A}} \mathbf{C}_{A'}$ and let $(\gamma_{A'}\colon \mathbf{C}_{A'} \to \mathbf{C} \mid A' \in \mathcal{A})$ be the corresponding limit cone. We define a mapping $\gamma\colon \mathbf{A} \to \mathbf{C}$ by

$$\gamma(a) := \gamma_{A'}\beta'_{A'}(a), \qquad a \in A$$

for any $A' \in \mathcal{A}$ with $a \in \tilde{A}'$. To see that this is independent of the choice of A', assume $a \in \tilde{A}_1, \tilde{A}_2$ with $A_1, A_2 \in \mathcal{A}$. Let $A_3 := A_1 \cup A_2$. Then we have

$$\gamma_{A_1}\beta'_{A_1}(a) = \gamma_{A_3}\gamma_{A_1,A_3}\beta'_{A_1}(a) = \gamma_{A_3}\beta'_{A_3}(a) = \gamma_{A_3}\gamma_{A_2,A_3}\beta'_{A_2}(a) = \gamma_{A_2}\beta'_{A_2}(a).$$

Since $\beta'_{A'}$ and $\gamma_{A'}$ are homomorphisms, γ is a homomorphism.

The homomorphism γ is injective. Indeed, let $a_1 \in \tilde{A}_1$ and $a_2 \in \tilde{A}_2$ with $a_1 \neq a_2$, and assume, to the contrary, $\gamma(a_1) = \gamma(a_2)$. That is, for some upper bound A_3 of A_1 and A_2 we have $\gamma_{A_1,A_3}\beta'_{A_1}(a_1) = \gamma_{A_2,A_3}\beta'_{A_2}(a_2)$, hence, $\beta'_{A_3}(a_1) = \beta'_{A_3}(a_2)$. But, by the claim above, $\beta'_{A_3} = \alpha'_{\tilde{A}_3}$ has a left inverse and therefore it is injective, a contradiction.

The homomorphism γ is surjective. Indeed, let $c \in C$, i.e., there is an $A' \in \mathcal{A}$ and a $c' \in C_{A'}$ such that $c = \gamma_{A'}(c')$. Let A'' be any finite subset of A properly containing A' and $a := \beta_{A'}(c')$. Then $a \in \tilde{A}''$ and we can calculate

$$\gamma(a) = \gamma_{A''}\beta'_{A''}(a) = \gamma_{A''}\beta'_{A''}\beta_{A'}(c') = \gamma_{A''}\gamma_{A',A''}(c') = \gamma_{A'}(c') = c.$$

It remains to show that γ is full. Let $\mathbf{C} \models \mathsf{r}(\bar{c})$, $\mathsf{r} \in \mathcal{R}$. That is, there is an $A' \in \mathcal{A}$ and a $\bar{c}' \in \mathbf{C}_{A'}$ such that $\bar{c} = \gamma_{A'}(\bar{c}')$ and $\mathbf{C}_{A'} \models \mathsf{r}(\bar{c}')$. Let A'' be any finite subset of A properly containing A' and $\bar{a} := \beta_{A'}(\bar{c}')$. Then $\mathbf{A} \models \mathsf{r}(\bar{a})$ and the same computation as above yields $\gamma(\bar{a}) = \bar{c}$. $\square$

2.4 Axiomatizing R-classes

Proposition 2.23(i) yields that for a class K of structures it holds $\mathrm{R}\,K \subseteq \mathrm{Mod}\,\mathrm{Th}_{\mathrm{pe}}\,K$. Under certain finiteness conditions equality holds.

Theorem 2.36. *Let $\mathbf{A}$ be a finite structure, and let K be a finite class of finite structures. If $\mathbf{A} \in \mathrm{Mod}\,\mathrm{Th}_{\mathrm{pe}}\,K$ then $\mathbf{A} \in \mathrm{R}\,K$.*

Proof. First, we consider the case when the type $\mathcal{R}$ is finite. We set

$$I := \{(\mathbf{B}, \alpha) \mid \mathbf{B} \in K,\ \alpha\colon \mathbf{A} \to \mathbf{B} \text{ is a homomorphism}\}.$$

Clearly, I is finite. For $i \in I$ we refer to the components of i by $\mathbf{B}_i$ and α_i. Let $\bar{a} = \langle a_0, \ldots, a_{m-1}\rangle$ be a tuple containing each element of A exactly once and let $\bar{x} = \langle x_0, \ldots, x_{m-1}\rangle$.

Claim (*). There is one $i^* \in I$ such that for all primitive-positive formulas $\varphi(\bar{x})$ we have $\mathbf{B}_{i^*} \models \varphi(\alpha_{i^*}(\bar{a}))$ implies $\mathbf{A} \models \varphi(\bar{a})$.

Proof of (*). Assume, to the contrary, that for each $i \in I$ we can find a formula $\varphi_i(\bar{x})$ with $\mathbf{B}_i \models \varphi_i(\alpha_i(\bar{a}))$ but $\mathbf{A} \not\models \varphi_i(\bar{a})$. Then the sentence

$$\varphi_{\mathbf{A},\bar{a}} \to \bigvee\nolimits_{i \in I} \varphi_i$$

is equivalent to a sentence in $\mathrm{Th}_{\mathrm{pe}}\, K$. It is satisfied by K but it is not satisfied by $\mathbf{A}$, a contradiction. This completes the proof of (*).

We set $\alpha' = \alpha_{i^*}$. The same reasoning as in the proof of Theorem 2.27, Claim (**), yields the existence of a homomorphism $\alpha\colon \mathbf{B}_{i^*} \to \mathbf{A}$ satisfying $\alpha\alpha' = \mathrm{id}_A$. Now, $(\alpha, \alpha')\colon \mathbf{B}_{i^*} \to \mathbf{A}$ is a retraction, hence, $\mathbf{A} \in \mathrm{R}\, K$.

Let $\mathcal{R}$ now be arbitrary. We modify the proof of Corollary 2.31. Consider the set of all finite types $\mathcal{R}' \subseteq \mathcal{R}$ ordered by inclusion. For every such $\mathcal{R}' \subseteq \mathcal{R}$ let $H_{\mathcal{R}'}$ be the set of all retractions $(\alpha, \alpha')\colon \mathbf{A}' \to \mathbf{B}'$, where $\mathbf{A}'$ is the $\mathcal{R}'$-reduct of $\mathbf{A}$ and $\mathbf{B}'$ is the $\mathcal{R}'$-reduct of a structure $\mathbf{B} \in K$. For $\mathcal{R}_2 \subseteq \mathcal{R}_1$ we define

$$\lambda_{\mathcal{R}_1,\mathcal{R}_2}\colon H_{\mathcal{R}_1} \to H_{\mathcal{R}_2}$$

to be the identical mapping on $H_{\mathcal{R}_1}$. This defines an inverse family of finite nonempty sets. Now, by a similar reasoning as in the proof of Corollary 2.31, we obtain a retraction of a structure $\mathbf{B} \in K$ to $\mathbf{A}$. $\square$

Examples 2.32 and 2.33 show that in Theorem 2.36 not both K and $\mathcal{R}$ can be infinite. The following example shows that finiteness conditions as in Theorem 2.27 are also not sufficient.

Example 2.37. Let $\mathcal{R}$ consist of one binary symbol r. Let $A = \{0, 1\}$ and define $\mathbf{A}$ by $\mathsf{r}^{\mathbf{A}} = \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 1,1\rangle\}$. Let K consist of structures $\mathbf{B}_2, \mathbf{B}_3, \ldots$ defined by $B_i = \{0, \ldots, i-1\}$ and

$$\mathsf{r}^{\mathbf{B}_i} = \{\langle j, j\rangle \mid j \in B_i\} \cup \{\langle j, j+1 \bmod i\rangle \mid j \in B_i\},$$

cf. Figure 2.2. Since there is no homomorphism of any B_j onto $\mathbf{A}$, we have $\mathbf{A} \notin \mathrm{R}\, K$.

Let φ_1 and φ_2 be existential-positive formulas such that $K \models \varphi_1 \to \varphi_2$. Assume $\mathbf{A} \not\models \varphi_1 \to \varphi_2$, i.e, there is an $\bar{a} \in A^m$ with $\mathbf{A} \models \varphi_1(\bar{a})$ and $\mathbf{A} \not\models \varphi_2(\bar{a})$. We choose i^* larger than the number of variables occurring in φ_2, and define $\alpha'\colon \mathbf{A} \to \mathbf{B}_{i^*}$ by $\alpha'(0) = 0$ and $\alpha'(1) = 1$. Since φ_1 is a positive formula,

$$\mathbf{B}_{i^*} \models \varphi_1(\alpha'(\bar{a})),$$

so $\mathbf{B}_{i^*} \models \varphi_2(\alpha'(\bar{a}))$. By the choice of i^*, we have a substructure $\mathbf{B}'_{i^*}$ of $\mathbf{B}_{i^*}$ with $i^* - 1$ elements, containing 0 and 1 such that $\mathbf{B}'_{i^*} \models \varphi_2(\alpha'(\bar{a}))$, cf. Figure 2.2. Obviously, there is a homomorphism $\alpha\colon \mathbf{B}'_{i^*} \to \mathbf{A}$ satisfying $\alpha\alpha' = \mathrm{id}_A$. This yields $\mathbf{A} \models \varphi_2(\alpha(\alpha'(\bar{a})))$, i.e., $\mathbf{A} \models \varphi_2(\bar{a})$. A contradiction. Hence, $\mathbf{A} \in \mathrm{Mod}\,\mathrm{Th}_{\mathrm{pe}}\, K$.

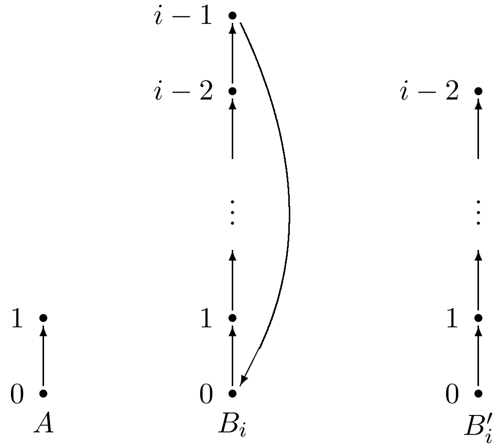

Figure 2.2: A counterexample

Chapter 3

Relational clones

In this chapter we examine the connection of relational equations satisfied by structures and properties of the clones of these structures, e.g. properties concerning the clone lattice $\mathbf{L}_A$.

3.1 Maltsev operators

A second approach of defining clones of relations is appropriate. We express the fact that the relation symbols occurring in a formula φ are among $\{\mathsf{r}_0, \ldots, \mathsf{r}_{l-1}\}$ by $\varphi(\mathsf{r}_0, \ldots, \mathsf{r}_{l-1})$, or, in combination with the indication of the occurring variables, by $\varphi(\mathsf{r}_0, \ldots, \mathsf{r}_{l-1}; x_0, \ldots, x_{m-1})$. Each formula $\varphi(\mathsf{r}_0, \ldots, \mathsf{r}_{l-1})$ defines a partial l-ary operation on $\mathrm{Rel}(A)$ by

$$\langle r_0, \ldots, r_{l-1} \rangle \mapsto \varphi^{(A, r_0, \ldots, r_{l-1})}.$$

Our second approach starts from operations on $\mathrm{Rel}(A)$, the so called Maltsev operators, and we derive a substitute for formulas. These operations are total and allow us to deal with algebras whose base sets consist of relations and whose base functions are the Maltsev operators. These operators have nothing to do with the terms and functions used in the classical Maltsev conditions for properties of congruence lattices.

Definition 3.1. We define the following operations on $\mathrm{Rel}(A)$.

- exchange of components

$$\zeta\colon \mathrm{Rel}(A) \to \mathrm{Rel}(A) \quad \zeta r := \{\langle a_0, \ldots, a_{m-1} \rangle \mid \langle a_1, a_2, \ldots, a_{m-1}, a_0 \rangle \in r\}$$
$$\tau\colon \mathrm{Rel}(A) \to \mathrm{Rel}(A) \quad \tau r := \{\langle a_0, \ldots, a_{m-1} \rangle \mid \langle a_1, a_0, a_2, \ldots, a_{m-1} \rangle \in r\}$$

- identify components $\triangle\colon \mathrm{Rel}(A) \to \mathrm{Rel}(A)$

$$\triangle r := \{\langle a_0, \ldots, a_{m-2} \rangle \mid \langle a_0, a_0, \ldots, a_{m-2} \rangle \in r\}$$

- composition $\circ\colon \mathrm{Rel}(A) \times \mathrm{Rel}(A) \to \mathrm{Rel}(A)$

$$\begin{aligned} r_1 \circ r_2 := \{&\langle a_0, \ldots, a_{m_1-2}, a'_0, \ldots, a'_{m_2-2} \rangle \mid \\ &(\exists a \in A)\ \langle a_0, \ldots, a_{m_1-2}, a \rangle \in r_1,\ \langle a, a'_0, \ldots, a'_{m_2-2} \rangle \in r_2\} \end{aligned}$$

For $r, r_1, r_2 \in \mathrm{Rel}^{(1)}(A)$ we set $\zeta r = \tau r = \triangle r = r$ and $r_1 \circ r_2 = \emptyset$. Finally, we define a constant operator on $\mathrm{Rel}(A)$, i.e., a single relation, by

$$d^{(3)}_{[1,23]} := \{\langle a_1, a_2, a_3\rangle \mid a_2 = a_3\}.$$

Definition 3.2. We abbreviate the sequence of operations $\zeta, \tau, \triangle, \circ, d^{(3)}_{[1,23]}$ by Mal. The algebra $\mathbf{Rel}(A) := (\mathrm{Rel}(A), \mathrm{Mal})$ is called the *full relation algebra* over A.

We motivate the next definition by an example. Assume we are given a type $\mathcal{R}$ consisting of one binary relation symbol r and the primitive-positive formula

$$\varphi(x_0, x_1) := (\exists y)\ \mathsf{r}(x_0, y) \wedge \mathsf{r}(x_1, y). \tag{3.1}$$

Obviously, for any structure $\mathbf{A}$

$$\varphi^{\mathbf{A}} = \mathsf{r}^{\mathbf{A}} \circ (\tau \mathsf{r}^{\mathbf{A}}).$$

We want to use "$\mathsf{r}\circ(\tau\mathsf{r})$" as a substitute for the formula (3.1). Actually, $\mathsf{r}\circ(\tau\mathsf{r})$ is a term built up from function symbols $\circ$ and τ and a variable r. It is natural to use the same notation for the symbols $\zeta, \tau, \triangle, \circ, d^{(3)}_{[1,23]}$ with arities $(1,1,1,2,0)$ and their interpretations in $\mathbf{Rel}(A)$.

Definition 3.3. Let $\mathcal{R}$ be a relational type. By $T_{\mathrm{Mal}}(\mathcal{R})$ we denote the set of all terms of type $(\zeta, \tau, \triangle, \circ, d^{(3)}_{[1,23]})$ with variables from $\mathcal{R}$. When we work with a fixed type $\mathcal{R}$, we write just T_{Mal} instead of $T_{\mathrm{Mal}}(\mathcal{R})$.

Let $\mathbf{A}$ be a structure of type $\mathcal{R}$, and let $p(\mathsf{r}_0, \dots, \mathsf{r}_{l-1}) \in T_{\mathrm{Mal}}(\mathcal{R})$. Then we define

$$p^{\mathbf{A}} := p^{\mathbf{Rel}(A)}(\mathsf{r}_0^{\mathbf{A}}, \dots, \mathsf{r}_{l-1}^{\mathbf{A}}).$$

The next proposition shows that terms in $T_{\mathrm{Mal}}(\mathcal{R})$ and primitive-positive formulas are in some sense equivalent.

Proposition 3.4 ([PK79]). *Let $\mathcal{R}$ be a type. For each primitive-positive formula φ of type $\mathcal{R}$ there exists a $p \in T_{\mathrm{Mal}}(\mathcal{R})$ such that it holds*

$$\varphi^{\mathbf{A}} = p^{\mathbf{A}}, \qquad \textit{for all structures } \mathbf{A} \textit{ of type } \mathcal{R}. \tag{*}$$

Vice versa, for each $p \in T_{\mathrm{Mal}}(\mathcal{R})$ there exists a primitive-positive formula φ such that (*) holds.

In [PK79] an effective procedure is described to obtain such a p for a given φ and vice versa. The correspondence of primitive-positive formulas and terms induces correspondences for the derived concepts of subuniverses and equations.

Corollary 3.5. *Let $R \subseteq \mathrm{Rel}(A)$. Then $\mathrm{Cln}\, R$ is the subuniverse of $\mathbf{Rel}(A)$ generated by R. Let $\mathbf{A}$ be a structure. Then $\mathrm{Cln}\,\mathbf{A} = \{p^{\mathbf{A}} \mid p \in T_{\mathrm{Mal}}(\mathcal{R})\}$.*

Corollary 3.6. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be structures. It holds* $\mathrm{Th}_{\mathrm{re}}\,\mathbf{A} \subseteq \mathrm{Th}_{\mathrm{re}}\,\mathbf{B}$ *if and only if for all* $p_1, p_2 \in T_{\mathrm{Mal}}$ *it holds*

$$p_1^{\mathbf{A}} = p_2^{\mathbf{A}} \quad \textit{implies} \quad p_1^{\mathbf{B}} = p_2^{\mathbf{B}}.$$

It holds $\mathrm{Th}_{\mathrm{re}}\,\mathbf{A} = \mathrm{Th}_{\mathrm{re}}\,\mathbf{B}$ *if and only if for all* $p_1, p_2 \in T_{\mathrm{Mal}}$ *it holds*

$$p_1^{\mathbf{A}} = p_2^{\mathbf{A}} \quad \textit{iff} \quad p_1^{\mathbf{B}} = p_2^{\mathbf{B}}.$$

Let $\mathbf{A}$ be a structure of type $\mathcal{R}$. Assume we are given a set of relational equations Σ such that $\mathrm{Mod}\,\Sigma = \mathrm{Mod}\,\mathrm{Th}_{\mathrm{re}}\,\mathbf{A}$. Let $\mathbf{A}'$ be a structure of type $\mathcal{R}'$, with $\mathrm{Cln}\,\mathbf{A} = \mathrm{Cln}\,\mathbf{A}'$. How can we derive from Σ a set Σ' such that $\mathrm{Mod}\,\Sigma' = \mathrm{Mod}\,\mathrm{Th}_{\mathrm{re}}\,\mathbf{A}'$? To simplify the presentation, we show this for $\mathcal{R}$ containing only one symbol r and $\mathcal{R}'$ containing only one symbol r'. In view of Proposition 3.4 and Corollary 3.6, we use relational equations of the form $p_1 \leftrightarrow p_2$, where $p_1, p_2 \in T_{\mathrm{Mal}}$. That is, $\mathbf{A} \models p_1 \leftrightarrow p_2$ if and only if $p_1^{\mathbf{A}} = p_2^{\mathbf{A}}$.

Lemma 3.7. *Let* $\mathcal{R}$, $\mathcal{R}'$, $\mathbf{A}$ *and* $\mathbf{A}'$ *be as above. Let* Σ *be a set of relational equations such that* $\mathrm{Mod}\,\Sigma = \mathrm{Mod}\,\mathrm{Th}_{\mathrm{re}}\,\mathbf{A}$*. By Corollary 3.5, there are* $p(\mathsf{r}') \in T_{\mathrm{Mal}}(\mathcal{R}')$ *and* $p'(\mathsf{r}) \in T_{\mathrm{Mal}}(\mathcal{R})$ *such that* $\mathsf{r}^{\mathbf{A}} = p^{\mathbf{A}'}$ *and* $\mathsf{r}'^{\mathbf{A}'} = p'^{\mathbf{A}}$*. We define* Σ' *to contain* $\mathsf{r}' \leftrightarrow p'(p(\mathsf{r}'))$ *and all relational equations*

$$p_1(p(\mathsf{r}')) \leftrightarrow p_2(p(\mathsf{r}')), \qquad p_1(\mathsf{r}) \leftrightarrow p_2(\mathsf{r}) \in \Sigma.$$

Then $\mathrm{Mod}\,\Sigma' = \mathrm{Mod}\,\mathrm{Th}_{\mathrm{re}}\,\mathbf{A}'$*.*

Proof. Obviously, $\mathbf{A}' \models \Sigma'$. Thus, $\mathrm{Mod}\,\Sigma' \supseteq \mathrm{Mod}\,\mathrm{Th}_{\mathrm{re}}\,\mathbf{A}'$.

To check the reverse inclusion, let $\mathbf{B}'$ be any structure of type $\mathcal{R}'$ with $\mathbf{B}' \models \Sigma'$. For any $p_1^*, p_2^* \in T_{\mathrm{Mal}}(\mathcal{R}')$ with $p_1^{*\mathbf{A}'} = p_2^{*\mathbf{A}'}$, we have to show $p_1^{*\mathbf{B}'} = p_2^{*\mathbf{B}'}$.

Consider the structure $\mathbf{B}$ of type $\mathcal{R}$ given by $\mathsf{r}^{\mathbf{B}} := p^{\mathbf{B}'}$. For all $p_1(\mathsf{r}) \leftrightarrow p_2(\mathsf{r}) \in \Sigma$ we have $(p_1(p))^{\mathbf{B}'} = (p_2(p))^{\mathbf{B}'}$, hence $\mathbf{B} \models \Sigma$, so $\mathbf{B} \models \mathrm{Th}_{\mathrm{re}}\,\mathbf{A}$. We conclude

$$\begin{aligned}
(p_1^*(p'))^{\mathbf{A}} = p_1^{*\mathbf{A}'} &= p_2^{*\mathbf{A}'} = (p_2^*(p'))^{\mathbf{A}}, \\
\implies \quad & (p_1^*(p'))^{\mathbf{B}} = (p_2^*(p'))^{\mathbf{B}}, \\
\implies \quad & (p_1^*(p'(p)))^{\mathbf{B}'} = (p_2^*(p'(p)))^{\mathbf{B}'}, \\
\implies \quad & p_1^{*\mathbf{B}'} = p_2^{*\mathbf{B}'}.
\end{aligned}$$

This completes the proof. □

3.2 Relational equations and clones

Now we are ready to study the connections between the relational equations satisfied by a structure $\mathbf{A}$ and the algebra $(\mathrm{Cln}\,\mathbf{A}, \mathrm{Mal})$.

Lemma 3.8. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be structures. Then* $\mathrm{Th}_{\mathrm{re}}\,\mathbf{A} \subseteq \mathrm{Th}_{\mathrm{re}}\,\mathbf{B}$ *(*$\mathrm{Th}_{\mathrm{re}}\,\mathbf{A} = \mathrm{Th}_{\mathrm{re}}\,\mathbf{B}$ *resp.) if and only if there is an onto homomorphism (isomorphism resp.)*

$$\gamma\colon (\mathrm{Cln}\,\mathbf{A}, \mathrm{Mal}) \to (\mathrm{Cln}\,\mathbf{B}, \mathrm{Mal})$$

with $\gamma(\mathsf{r}^{\mathbf{A}}) = \mathsf{r}^{\mathbf{B}}$ *for all* $\mathsf{r} \in \mathcal{R}$.

By Corollary 3.5, $\{\mathsf{r}^{\mathbf{A}} \mid \mathsf{r} \in \mathcal{R}\}$ *generates* $(\mathrm{Cln}\,\mathbf{A}, \mathrm{Mal})$*. Thus, any such homomorphism (isomorphism resp.)* γ *even satisfies* $\gamma(p^{\mathbf{A}}) = p^{\mathbf{B}}$, $p \in T_{\mathrm{Mal}}(\mathcal{R})$.

Proof. "$\Longrightarrow$". Let $\mathbb{T} = (T_{\mathrm{Mal}}(\mathcal{R}), \mathrm{Mal})$ be the term algebra with variables $\mathcal{R}$ of type $(\zeta, \tau, \triangle, \circ, d^{(3)}_{[1,23]})$. We define

$$\gamma_{\mathbf{A}}\colon \mathbb{T} \to (\mathrm{Cln}\,\mathbf{A}, \mathrm{Mal})$$

to be the surjective homomorphism given by $\gamma_{\mathbf{A}}(p) := p^{\mathbf{A}}$, and define

$$\gamma_{\mathbf{B}}\colon \mathbb{T} \to (\mathrm{Cln}\,\mathbf{B}, \mathrm{Mal})$$

to be the surjective homomorphism given by $\gamma_{\mathbf{B}}(p) := p^{\mathbf{B}}$. Corollary 3.6 and $\mathrm{Th}_{\mathrm{re}}\,\mathbf{A} \subseteq \mathrm{Th}_{\mathrm{re}}\,\mathbf{B}$ imply that for the kernels of $\gamma_{\mathbf{A}}$ and $\gamma_{\mathbf{B}}$ it holds $\ker\gamma_{\mathbf{A}} \subseteq \ker\gamma_{\mathbf{B}}$. Respectively, Corollary 3.6 and $\mathrm{Th}_{\mathrm{re}}\,\mathbf{A} = \mathrm{Th}_{\mathrm{re}}\,\mathbf{B}$ imply $\ker\gamma_{\mathbf{A}} = \ker\gamma_{\mathbf{B}}$. In both cases, the well known "second isomorphism theorem" states the existence of a γ with the required properties.

"$\Longleftarrow$". Immediate by Corollary 3.6. ☐

Lemma 3.8 suggests the following definition.

Definition 3.9. Let $R_1 \subseteq \mathrm{Rel}(A)$ and $R_2 \subseteq \mathrm{Rel}(B)$ be clones. We define $R_1 \to R_2$ ($R_1 \leftrightarrow R_2$ resp.) if there is an onto homomorphism (isomorphism resp.)

$$\gamma\colon (R_1, \mathrm{Mal}) \to (R_2, \mathrm{Mal}).$$

It is well known that any such homomorphism (isomorphism resp.) γ preserves arity, i.e., $\mathrm{ar}(r) = \mathrm{ar}(\gamma(r))$, $r \in R_1$. Since relational clones are mostly given by a generating structure, we need to make precise the connection of the notions defined above and the relational equations satisfied by generating structures. Proposition 3.11 below states that, to a large extend, this is independent of the choice of these structures.

Remark 3.10. In what follows, we frequently encounter nonindexed structures, i.e., base sets A together with a set of relations R, not belonging to a prescribed type. We can formally assign a symbol to each relation in R. Thus, such nonindexed structures fit into our notion of structures. When we do not use concepts referring to the type (e.g. constructions of structures), we do not mention this type explicitly. The same applies to algebras.

Proposition 3.11. *Let* $R_1 \subseteq \mathrm{Rel}(A)$ *and* $R_2 \subseteq \mathrm{Rel}(B)$ *be clones. Then the following are equivalent:*

(i) $R_1 \to R_2$ *(*$R_1 \leftrightarrow R_2$ *resp.),*

(ii) *there exist structures* $\mathbf{A}$ *and* $\mathbf{B}$ *such that* $\mathrm{Cln}\,\mathbf{A} = R_1$, $\mathrm{Cln}\,\mathbf{B} = R_2$ *and* $\mathrm{Th}_{\mathrm{re}}\,\mathbf{A} \subseteq \mathrm{Th}_{\mathrm{re}}\,\mathbf{B}$ *(*$\mathrm{Th}_{\mathrm{re}}\,\mathbf{A} = \mathrm{Th}_{\mathrm{re}}\,\mathbf{B}$ *resp.),*

(iii) *for any structure* $\mathbf{A}$ *with* $\mathrm{Cln}\,\mathbf{A} = R_1$ *there exists a structure* $\mathbf{B}$ *such that* $\mathrm{Cln}\,\mathbf{B} = R_2$ *and* $\mathrm{Th}_{\mathrm{re}}\,\mathbf{A} \subseteq \mathrm{Th}_{\mathrm{re}}\,\mathbf{B}$ *(*$\mathrm{Th}_{\mathrm{re}}\,\mathbf{A} = \mathrm{Th}_{\mathrm{re}}\,\mathbf{B}$ *resp.).*

Proof. Since $\mathrm{Cln}(A, R_1) = R_1$, we have "(iii)$\Longrightarrow$(ii)".

Lemma 3.8 yields "(ii)$\Longrightarrow$(i)".

(i)$\Longrightarrow$(iii). Let $\mathbf{A}$ be any structure with $\mathrm{Cln}\,\mathbf{A} = R_1$. Let $\gamma\colon (R_1, \mathrm{Mal}) \to (R_2, \mathrm{Mal})$ be an onto homomorphism. Define $\mathbf{B}$ by $\mathsf{r}^{\mathbf{B}} := \gamma(\mathsf{r}^{\mathbf{A}})$, $\mathsf{r} \in \mathcal{R}$. By Corollary 3.5 and since γ is onto, we have $\mathrm{Cln}\,\mathbf{B} = R_2$. Now Lemma 3.8 yields $\mathrm{Th}_{\mathrm{re}}\,\mathbf{A} \subseteq \mathrm{Th}_{\mathrm{re}}\,\mathbf{B}$ ($\mathrm{Th}_{\mathrm{re}}\,\mathbf{A} = \mathrm{Th}_{\mathrm{re}}\,\mathbf{B}$ resp.). $\square$

Proposition 3.12. *Let A be finite, and let $R_1 \subseteq \mathrm{Rel}(A)$ and $R_2 \subseteq \mathrm{Rel}(B)$ be clones.*

(i) If $\mathbf{A}$ *and* $\mathbf{A}'$ *are such that* $\mathrm{Cln}\,\mathbf{A} = \mathrm{Cln}\,\mathbf{A}' = R_1$ *and* $\mathrm{Th}_{\mathrm{re}}\,\mathbf{A} \subseteq \mathrm{Th}_{\mathrm{re}}\,\mathbf{A}'$, *then* $\mathrm{Th}_{\mathrm{re}}\,\mathbf{A} = \mathrm{Th}_{\mathrm{re}}\,\mathbf{A}'$.

(ii) If $R_1 \to R_2 \to R_1$, then $R_1 \leftrightarrow R_2$.

Proof. (i). By Corollary 3.5 we have $\mathrm{Cln}\,\mathbf{A} = \{p^{\mathbf{A}} \mid p \in T_{\mathrm{Mal}}(\mathcal{R})\}$ and $\mathrm{Cln}\,\mathbf{A}' = \{p^{\mathbf{A}'} \mid p \in T_{\mathrm{Mal}}(\mathcal{R})\}$. By Lemma 3.8, there is an onto homomorphism

$$\gamma\colon (R_1, \mathrm{Mal}) \to (R_1, \mathrm{Mal})$$

with $\gamma(p^{\mathbf{A}}) = p^{\mathbf{A}'}$ for all $p \in T_{\mathrm{Mal}}(\mathcal{R})$. Since γ preserves arity, and the set $R_1^{(m)}$ is finite for a fixed arity m, γ is a bijection and, in turn, an isomorphism. Lemma 3.8 yields $\mathrm{Th}_{\mathrm{re}}\,\mathbf{A} = \mathrm{Th}_{\mathrm{re}}\,\mathbf{A}'$.

(ii). By Lemma 3.8 there are onto homomorphisms

$$\gamma_1\colon (R_1, \mathrm{Mal}) \to (R_2, \mathrm{Mal}) \qquad \text{and}$$
$$\gamma_2\colon (R_2, \mathrm{Mal}) \to (R_1, \mathrm{Mal}).$$

Then $\gamma_1\gamma_2$ is an onto endomorphism of (R_1, Mal). Since $\gamma_1\gamma_2$ preserves arity, and the set $R_1^{(m)}$ is finite for a fixed arity m, $\gamma_1\gamma_2$ is bijective, hence γ_1 is bijective and, in turn, an isomorphism. Thus, $R_1 \leftrightarrow R_2$. $\square$

Problem 3.13. Does Proposition 3.12 hold for infinite A?

By $[D(A), R]_{\mathbf{L}_A}$ we denote the interval $[D(A), R]$ in the lattice $\mathbf{L}_A$. Next we relate "$R_1 \to R_2$" and "$R_1 \leftrightarrow R_2$" with properties of the intervals $[D(A), R_1]_{\mathbf{L}_A}$ and $[D(B), R_2]_{\mathbf{L}_B}$. This is an immediate consequence of Corollary 3.5 and the fact that for a clone R the interval $[D(A), R]$ in $\mathbf{L}_A$ is the subalgebra lattice of (R, Mal).

Lemma 3.14. *Let $R_1 \subseteq \mathrm{Rel}(A)$ and $R_2 \subseteq \mathrm{Rel}(B)$ be clones. If $R_1 \to R_2$ then there is a mapping*

$$\gamma\colon [D(A), R_1]_{\mathbf{L}_A} \to [D(B), R_2]_{\mathbf{L}_B}$$

preserving join, and a mapping

$$\gamma' \colon [D(B), R_2]_{\mathbf{L}_B} \to [D(A), R_1]_{\mathbf{L}_A}$$

preserving meet. Further, $\gamma\gamma' = \mathrm{id}_{[D(A),R_1]_{\mathbf{L}_A}}$ *and for all* $R \in [D(A), R_1]_{\mathbf{L}_A}$ *we have* $R \to \gamma(R)$*.*

If $R_1 \leftrightarrow R_2$ *then* γ *is an isomorphism and for all* $R \in [D(A), R_1]_{\mathbf{L}_A}$ *we have* $R \leftrightarrow \gamma(R)$*.*

Chapter 4

Connections with algebras

The concept of a clone of relations is closely related to the concept of a clone of functions. We extend this connection. First, let us make precise a correspondence between relational structures and algebras.

Definition 4.1. Let $\mathbb{A}$ be an algebra, and let $\mathbf{A}$ be a structure with the same finite base set A. We write $\mathbb{A} \sim \mathbf{A}$ if $\operatorname{Inv} \mathbb{A} = \operatorname{Cln} \mathbf{A}$ or, equivalently, $\operatorname{Cln} \mathbb{A} = \operatorname{Pol} \mathbf{A}$.

When we change from structures $\mathbf{A}$ to algebras $\mathbb{A} \sim \mathbf{A}$, how the concepts for structures of the foregoing sections translate? This chapter is devoted to fill in the following dictionary.

retract	idempotent image
power	matrix power
$\mathrm{Th}_{\mathrm{re}}$-equivalent	categorically equivalent

Moreover, in Theorem 4.11 we give the counterpart of Theorem 2.28.

4.1 Constructions of algebras

We relate the formation of retracts and powers of structures to certain constructions of algebras. The facts given in Corollaries 4.4 and 4.7 can be found in another form in [DL01, Lar00, Zád97]. Actually, we start with constructions for sets of relations (sets of functions resp.), which is a little more general. These constructions can be regarded as dual versions of constructions analyzed in [PK79].

Definition 4.2. Let $R \subseteq \operatorname{Rel} A$, let $F \subseteq \operatorname{Func} A$ and let $\alpha\colon A \to A$ be an idempotent mapping. We denote

$$\alpha(R) := \{\alpha(r) \mid r \in R\},$$
$$F(\alpha) := \{\alpha f{\restriction}_{\alpha(A)} \mid f \in F\}.$$

Let $\mathbb{A} = (A, F)$ be an algebra, and let α be a unary and idempotent term function of $\mathbb{A}$, i.e., $\alpha \in \operatorname{Cln}^{(1)} \mathbb{A}$ and idempotent. Then the *idempotent image* $\mathbb{A}(\alpha)$ of $\mathbb{A}$ is the algebra $(\alpha(A), (\operatorname{Cln} F)(\alpha))$.

Note that, for a structure $\mathbf{A} = (A, R)$ and an idempotent endomorphism α of $\mathbf{A}$, we have that $(\alpha(A), \alpha(R))$ is the retract $\alpha(\mathbf{A})$ (cf. Remark 2.7). To be precise, $(\alpha(A), \alpha(R))$ means the structure with base set $\alpha(A)$ and base relations

$$\mathsf{r}^{(\alpha(A),\alpha(R))} := \alpha(\mathsf{r}^{\mathbf{A}}), \qquad \mathsf{r} \in \mathcal{R}.$$

Since often only the term functions of $\mathbb{A}(\alpha)$ are considered, it is common usage to define $\mathbb{A}(\alpha)$ to be "the algebra with term functions $(\operatorname{Cln} F)(\alpha)$".

Lemma 4.3. *Let A be finite, let $R \subseteq \operatorname{Rel}(A)$ and $F \subseteq \operatorname{Func}(A)$ such that $\operatorname{Pol}_A R = \operatorname{Cln}_A F$ or, equivalently, $\operatorname{Inv}_A F = \operatorname{Cln}_A R$. Let α be an idempotent mapping on A with $\alpha \in \operatorname{End}_A R = \operatorname{Cln}_A^{(1)} F$. Then the following statements hold*

$$\operatorname{Cln}_{\alpha(A)}(F(\alpha)) \subseteq (\operatorname{Cln}_A F)(\alpha), \tag{4.1}$$

$$\alpha(\operatorname{Inv}_A F) \subseteq \operatorname{Inv}_{\alpha(A)}(F(\alpha)), \tag{4.2}$$

$$\operatorname{Cln}_{\alpha(A)}(\alpha(R)) = \alpha(\operatorname{Cln}_A R), \tag{4.3}$$

$$(\operatorname{Pol}_A R)(\alpha) = \operatorname{Pol}_{\alpha(A)}(\alpha(R)). \tag{4.4}$$

If, in addition, F is a clone then in (4.1) *and* (4.2) *equality holds.*

Proof. Let $f \in F$ and $r \in R$. Then both f and α preserve r, and Definition 1.5 yields that $\alpha f\restriction_{\alpha(A)}$ preserves $\alpha(r)$. Thus, $F(\alpha) \subseteq \operatorname{Pol}_{\alpha(A)}(\alpha(R))$. By Theorem 1.6 we obtain

$$\begin{aligned}
(\operatorname{Cln}_A F)(\alpha) &\subseteq \operatorname{Pol}_{\alpha(A)}(\alpha(R)),\\
\operatorname{Cln}_{\alpha(A)}(F(\alpha)) &\subseteq \operatorname{Pol}_{\alpha(A)}(\alpha(R)),\\
\alpha(\operatorname{Cln}_A R) &\subseteq \operatorname{Inv}_{\alpha(A)}(F(\alpha)),\\
\operatorname{Cln}_{\alpha(A)}(\alpha(R)) &\subseteq \operatorname{Inv}_{\alpha(A)}(F(\alpha)).
\end{aligned}$$

Since α is a retraction, we obtain, by Lemma 2.15, that for all primitive-positive formulas φ it holds $\alpha(\varphi^{(A,R)}) = \varphi^{\alpha(A,R)}$. Hence,

$$\alpha(\operatorname{Cln}_A R) = \operatorname{Cln}_{\alpha(A)}(\alpha(R)).$$

Let $f \in \operatorname{Pol}_{\alpha(A)}(\alpha(R))$. Define $\tilde{f} := f\alpha$. For any $r \in R$, we have f preserves $\alpha(r)$, and, by Definition 1.5, $\tilde{f}$ preserves r. Hence, $\tilde{f} \in \operatorname{Pol}_A R$. Since $\alpha\tilde{f}\restriction_{\alpha(A)} = f$, we obtain

$$\operatorname{Pol}_{\alpha(A)}(\alpha(R)) \subseteq (\operatorname{Pol}_A R)(\alpha).$$

This completes the proof of (4.1)–(4.4).

Let now F be a clone, i.e., $\operatorname{Cln}_A F = F$. Hence,

$$(\operatorname{Cln}_A F)(\alpha) = F(\alpha) \subseteq \operatorname{Cln}_{\alpha(A)}(F(\alpha)).$$

Using Theorem 1.6 we obtain

$$\begin{aligned}
\operatorname{Inv}_{\alpha(A)}(F(\alpha)) &= \operatorname{Inv}_{\alpha(A)}((\operatorname{Pol}_A R)(\alpha)) = \operatorname{Inv}_{\alpha(A)}(\operatorname{Pol}_{\alpha(A)}(\alpha(R)))\\
&= \operatorname{Cln}_{\alpha(A)}(\alpha(R)) = \alpha(\operatorname{Cln}_A R) = \alpha(\operatorname{Inv}_A F).
\end{aligned}$$

□

An example that we cannot omit the condition "F is a clone" is the following. Let $A = \{0, \dots, 3\}$ and $F = \{f, g\}$, where f and g is given by

x	0	1	2	3
$f(x)$	2	3	3	2
$g(x)$	0	1	0	1 .

Furthermore, let $\alpha := g$, and let $R := \operatorname{Inv}_A F$. Obviously, A, F, R and α satisfy the assumptions of Lemma 4.3. We have $\alpha(A) = \{0, 1\}$ and $F(\alpha) = \{\mathrm{id}_{\alpha(A)}\}$. The mapping $\alpha f f\restriction_{\alpha(A)} \in (\operatorname{Cln}_A F)(\alpha)$ interchanges 0 and 1. Hence,

$$\operatorname{Cln}_{\alpha(A)}(F(\alpha)) \neq (\operatorname{Cln}_A F)(\alpha).$$

This easily implies $\alpha(\operatorname{Inv}_A F) \neq \operatorname{Inv}_{\alpha(A)}(F(\alpha))$.

Now we are ready to state that the formation idempotent images and retractions are corresponding constructions.

Corollary 4.4. *Let A be finite, let $\mathbb{A}$ be an algebra and let $\mathbf{A}$ be a structure such that $\mathbb{A} \sim \mathbf{A}$. Let $\alpha \in \operatorname{Cln}^{(1)} \mathbb{A} = \operatorname{End} \mathbf{A}$ be an idempotent mapping. Then*

$$\mathbb{A}(\alpha) \sim \alpha(\mathbf{A}).$$

In a completely analogous way, we examine another pair of constructions.

Definition 4.5. Let $R \subseteq \operatorname{Rel} A$, let $F \subseteq \operatorname{Func} A$ and let $m \in \mathbb{N}_+$. We denote

$$R^{[m]} := \{r^{[m]} \mid r \in R\},$$

where $r^{[m]} \in \operatorname{Rel}(A^m)$ is defined by

$$\langle a_0, \dots, a_{\operatorname{ar}(r)-1}\rangle \in r^{[m]} \quad \text{iff} \quad \langle a_0(i), \dots, a_{\operatorname{ar}(r)-1}(i)\rangle \in r \quad \text{for all } i \in \{0, \dots, m-1\}.$$

We denote

$$F^{[m]} := \{[f_0, \dots, f_{m-1}] \mid f_0, \dots, f_{m-1} \in F^{(nm)},\ n \in \mathbb{N}_+\},$$

where $[f_0, \dots, f_{m-1}] \in \operatorname{Func}^{(n)}(A^m)$ is defined by

$$[f_0, \dots, f_{m-1}](a_0, \dots, a_{n-1})(i) := f_i(a_0(0), \dots, a_0(m-1), \dots, a_{n-1}(0), \dots, a_{n-1}(m-1)), \quad i \in \{0, \dots, m-1\}.$$

Let $\mathbb{A} = (A, F)$ be an algebra and let $m \in \mathbb{N}_+$. Then the m-th *matrix power* $\mathbb{A}^{[m]}$ of $\mathbb{A}$ is the algebra $(A^m, (\operatorname{Cln} F)^{[m]})$.

Note that, for a structure $\mathbf{A} = (A, R)$, we have that $(A^m, R^{[m]})$ is the power $\mathbf{A}^m$. To be precise, $(A^m, R^{[m]})$ means the structure with base set A^m and base relations

$$\mathsf{r}^{(A^m, R^{[m]})} := (\mathsf{r}^{\mathbf{A}})^{[m]}, \qquad \mathsf{r} \in \mathcal{R}.$$

Since often only the term functions of $\mathbb{A}^{[m]}$ are considered, it is common usage to define $\mathbb{A}^{[m]}$ to be "the algebra with term functions $(\operatorname{Cln} F)^{[m]}$".

Lemma 4.6. *Let A be finite, let $R \subseteq \operatorname{Rel}(A)$ and $F \subseteq \operatorname{Func}(A)$ such that $\operatorname{Pol}_A R = \operatorname{Cln}_A F$ or, equivalently, $\operatorname{Inv}_A F = \operatorname{Cln}_A R$. Let $m \in \mathbb{N}_+$. Then the following statements hold*

$$\operatorname{Cln}_{A^m}(F^{[m]}) \subseteq (\operatorname{Cln}_A F)^{[m]}, \tag{4.5}$$
$$(\operatorname{Inv}_A F)^{[m]} \subseteq \operatorname{Inv}_{A^m}(F^{[m]}), \tag{4.6}$$
$$\operatorname{Cln}_{A^m}(R^{[m]}) = (\operatorname{Cln}_A R)^{[m]}, \tag{4.7}$$
$$(\operatorname{Pol}_A R)^{[m]} = \operatorname{Pol}_{A^m}(R^{[m]}). \tag{4.8}$$

If, in addition, F is a clone then in (4.5) *and* (4.6) *equality holds.*

Proof. The proof follows the same pattern as the proof of Lemma 4.3. Let $f_0, \dots, f_{m-1} \in F^{(nm)}$, $n \in \mathbb{N}_+$, and $r \in R$. Then all f_i preserve r, and Definition 1.5 yields that $[f_0, \dots, f_{m-1}]$ preserves $r^{[m]}$. Thus, $F^{[m]} \subseteq \operatorname{Pol}_{A^m}(R^{[m]})$. By Theorem 1.6 we obtain

$$\begin{aligned}
(\operatorname{Cln}_A F)^{[m]} &\subseteq \operatorname{Pol}_{A^m}(R^{[m]}),\\
\operatorname{Cln}_{A^m}(F^{[m]}) &\subseteq \operatorname{Pol}_{A^m}(R^{[m]}),\\
(\operatorname{Cln}_A R)^{[m]} &\subseteq \operatorname{Inv}_{A^m}(F^{[m]}),\\
\operatorname{Cln}_{A^m}(R^{[m]}) &\subseteq \operatorname{Inv}_{A^m}(F^{[m]}).
\end{aligned}$$

By Lemma 2.16, we obtain that for all primitive-positive formulas φ it holds $(\varphi^{(A,R)})^{[m]} = \varphi^{(A,R)^m}$. Hence,

$$(\operatorname{Cln}_A R)^{[m]} = \operatorname{Cln}_{A^m}(R^{[m]}).$$

Let $f \in \operatorname{Pol}^{(n)}_{A^m}(R^{[m]})$. For $i \in \{0, \dots, m-1\}$, define $\tilde{f}_i \in \operatorname{Func}^{(nm)}(A)$ by

$$\tilde{f}_i(a_0, \dots, a_{nm-1}) := f((a_0, \dots, a_{m-1}), \dots, (a_{(n-1)m}, \dots, a_{nm-1}))(i).$$

For any $r \in R$, we have f preserves $r^{[m]}$, and, by Definition 1.5, all $\tilde{f}_i$ preserve r. Hence, $\tilde{f}_i \in \operatorname{Pol}_A R$. Since $[f_0, \dots, f_{m-1}] = f$, we obtain

$$\operatorname{Pol}_{A^m}(R^{[m]}) \subseteq (\operatorname{Pol}_A R)^{[m]}.$$

This completes the proof of (4.5)–(4.8).

Let now F be a clone, i.e. $\operatorname{Cln}_A F = F$. Hence,

$$(\operatorname{Cln}_A F)^{[m]} = F^{[m]} \subseteq \operatorname{Cln}_{A^m}(F^{[m]}).$$

Using Theorem 1.6 we obtain

$$\begin{aligned}
\operatorname{Inv}_{A^m}(F^{[m]}) &= \operatorname{Inv}_{A^m}((\operatorname{Pol}_A R)^{[m]}) = \operatorname{Inv}_{A^m}(\operatorname{Pol}_{A^m}(R^{[m]}))\\
&= \operatorname{Cln}_{A^m}(R^{[m]}) = (\operatorname{Cln}_A R)^{[m]} = (\operatorname{Inv}_A F)^{[m]}.
\end{aligned}$$

∎

It is trivial that we cannot omit the condition "F is a clone" in Lemma 4.6: For any F we can add fictive arguments to the functions of F such that $F^{(nm)} = \emptyset$ for all $n \in \mathbb{N}_+$, hence $F^{[m]} = \emptyset$, while keeping $\operatorname{Cln} F$ unchanged.

Now we are ready to state that the formation matrix powers and powers are corresponding constructions.

Corollary 4.7. *Let A be finite, let $\mathbb{A}$ be an algebra, and let $\mathbf{A}$ be a structure such that $\mathbb{A} \sim \mathbf{A}$. Let $m \in \mathbb{N}_+$. Then*

$$\mathbb{A}^{[m]} \sim \mathbf{A}^m.$$

4.2 Categorically equivalent algebras

We consider the connection between structures satisfying the same relational equations and categorically equivalent algebras.

Definition 4.8. We consider varieties of algebras as categories, i.e., the objects are the algebras of the variety and the morphisms are the homomorphisms between them. Let V be a variety of algebras. For two algebras $\mathbb{A}_1$ and $\mathbb{A}_2$ in V we denote by $\hom_V(\mathbb{A}_1, \mathbb{A}_2)$ the set of homomorphisms from $\mathbb{A}_1$ to $\mathbb{A}_2$. By $\mathbb{A}_1 \simeq \mathbb{A}_2$ we denote that the algebras $\mathbb{A}_1$ and $\mathbb{A}_2$ are isomorphic.

Let V and W be varieties of algebras. A functor $\mu\colon V \to W$ is a *categorical equivalence* if μ maps $\hom_V(\mathbb{A}_1, \mathbb{A}_2)$ bijectively to $\hom_W(\mu(\mathbb{A}_1), \mu(\mathbb{A}_2))$ for all $\mathbb{A}_1, \mathbb{A}_2 \in V$, and for all $\mathbb{B} \in W$ there is an $\mathbb{A} \in V$ with $\mu(\mathbb{A}) \simeq \mathbb{B}$.

We call two algebras $\mathbb{A}$ and $\mathbb{B}$ *categorically equivalent*, $\mathbb{A} \equiv_{\mathrm{c}} \mathbb{B}$, if there is a categorical equivalence $\mu\colon\ \mathrm{V}(\mathbb{A}) \to \mathrm{V}(\mathbb{B})$ with $\mu(\mathbb{A}) = \mathbb{B}$.

We call two algebras $\mathbb{A}$ and $\mathbb{B}$ *term equivalent*, $\mathbb{A} \equiv_{\mathrm{t}} \mathbb{B}$, if they have the same base set and it holds $\operatorname{Cln} \mathbb{A} = \operatorname{Cln} \mathbb{B}$.

Finally, two algebras $\mathbb{A}$ and $\mathbb{B}$ are *equivalent* or *weakly isomorphic*, $\mathbb{A} \equiv \mathbb{B}$, if there is an algebra $\mathbb{B}'$ such that

$$\mathbb{A} \equiv_{\mathrm{t}} \mathbb{B}' \simeq \mathbb{B}.$$

A list of properties of algebras preserved under categorical equivalence is given in [McK96]. Examples of such properties are primality [Hu69], various generalizations of primality, and finiteness.

The concept of categorical equivalence is connected to our notion of relational equations by the following theorem.

Theorem 4.9 ([DL01, Lüd93]). *Let $\mathbb{A}$ and $\mathbb{B}$ be finite algebras. Then it holds $\mathbb{A} \equiv_{\mathrm{c}} \mathbb{B}$ if and only if the algebras $(\operatorname{Inv} \mathbb{A}, \mathrm{Mal})$ and $(\operatorname{Inv} \mathbb{B}, \mathrm{Mal})$ are isomorphic.*

By Proposition 3.11, we obtain:

Corollary 4.10. *Let A and B be finite, and let $\mathbb{A}$ and $\mathbb{B}$ be algebras. Then it holds $\mathbb{A} \equiv_{\mathrm{c}} \mathbb{B}$ if and only if there are structures $\mathbf{A}$ and $\mathbf{B}$ of the same type such that $\mathbb{A} \sim \mathbf{A}$, $\mathbb{B} \sim \mathbf{B}$ and $\operatorname{Th}_{\mathrm{re}} \mathbf{A} = \operatorname{Th}_{\mathrm{re}} \mathbf{B}$.*

Now we are ready to translate Theorem 2.28. In [McK96] R. McKenzie showed that any algebra categorically equivalent to a given algebra $\mathbb{A}$ is equivalent to an idempotent image of a matrix power of $\mathbb{A}$ (cf. Theorem 4.11 below). In [Zád97] L. Zádori concluded from this that two algebras $\mathbb{A}$ and $\mathbb{B}$ are categorically equivalent if and only if there are structures $\mathbf{A} \sim \mathbb{A}$ and $\mathbf{B} \sim \mathbb{B}$ such that $\mathrm{R}\,\mathrm{P}_{\mathrm{fin}}\,\mathbf{A} = \mathrm{R}\,\mathrm{P}_{\mathrm{fin}}\,\mathbf{B}$. Further, using Theorem 4.9, K. Denecke and O. Lüders [DL01] obtained from McKenzie's result that for two finite algebras $\mathbb{A}$ and $\mathbb{B}$ it holds $(\mathrm{Inv}\,\mathbb{A}, \mathrm{Mal}) \simeq (\mathrm{Inv}\,\mathbb{B}, \mathrm{Mal})$ if and only if $\mathrm{R}\,\mathrm{P}_{\mathrm{fin}}(A, \mathrm{Inv}\,\mathbb{A}) = \mathrm{R}\,\mathrm{P}_{\mathrm{fin}}(B, \mathrm{Inv}\,\mathbb{B})$. Taking all this and Lemma 3.8 we could obtain a part of Theorem 2.28. Our approach is completely different. We obtained Theorem 2.28 by a model-theoretic approach, and prove Theorem 4.11 from this and the results gathered in this chapter.

Theorem 4.11 ([McK96]). *Let $\mathbb{A}$ and $\mathbb{B}$ be finite algebras. Then the following are equivalent.*

(i) $\mathbb{A} \equiv_{\mathrm{c}} \mathbb{B}$.

(ii) *There are $m, m' \in \mathbb{N}_+$, a unary, idempotent $\alpha \in \mathrm{Cln}\,\mathbb{A}^{[m]}$ and a unary, idempotent $\beta \in \mathrm{Cln}\,\mathbb{B}^{[m']}$ such that*

$$\mathbb{B} \equiv \mathbb{A}^{[m]}(\alpha),$$
$$\mathbb{A} \equiv \mathbb{B}^{[m']}(\beta).$$

(iii) *There are $m \in \mathbb{N}_+$ and a unary, idempotent $\alpha \in \mathrm{Cln}\,\mathbb{A}^{[m]}$ such that α is invertible by $\mathrm{Cln}\,\mathbb{A}^{[m]}$ and*

$$\mathbb{B} \equiv \mathbb{A}^{[m]}(\alpha).$$

The original version in [McK96] is the equivalence "(i)$\Longleftrightarrow$(iii)" for arbitrary (not only finite) algebras.

Proof. "(i)$\Longrightarrow$(ii), (iii)". Let $\mathbb{A} \equiv_{\mathrm{c}} \mathbb{B}$. By Corollary 4.10, there are structures $\mathbf{A}$ and $\mathbf{B}$ with $\mathbb{A} \sim \mathbf{A}$ and $\mathbb{B} \sim \mathbf{B}$ such that

$$\mathrm{Th}_{\mathrm{re}}\,\mathbf{A} = \mathrm{Th}_{\mathrm{re}}\,\mathbf{B}.$$

Theorem 2.28 yields that

$$\mathbf{B} \in \mathrm{R}\,\mathrm{P}_{\mathrm{fin}}\,\mathbf{A} \qquad \text{and} \qquad \mathbf{A} \in \mathrm{R}\,\mathrm{P}_{\mathrm{fin}}\,\mathbf{B}.$$

By Remark 2.7, from $\mathbf{B} \in \mathrm{R}\,\mathrm{P}_{\mathrm{fin}}\,\mathbf{A}$ it follows that there is an idempotent endomorphism α of a power $\mathbf{A}^m$ such that $\mathbf{B}$ is isomorphic to $\mathbf{B}' := \alpha(\mathbf{A}^m)$. Renaming the elements of B according to this isomorphism gives an algebra $\mathbb{B}'$ isomorphic to $\mathbb{B}$ such that $\mathbb{B}' \sim \mathbf{B}'$. We have

$$\mathrm{Cln}\,\mathbb{B}' = \mathrm{Pol}\,\mathbf{B}' = \mathrm{Pol}(\alpha(\mathbf{A}^m)).$$

By Corollaries 4.4 and 4.7,

$$\mathrm{Pol}(\alpha(\mathbf{A}^m)) = \mathrm{Cln}(\mathbb{A}^{[m]}(\alpha)),$$

hence $\mathbb{B}' \equiv_{\mathrm{t}} \mathbb{A}^{[m]}(\alpha)$, so $\mathbb{B} \equiv \mathbb{A}^{[m]}(\alpha)$. Analogously, $\mathbf{A} \in \mathrm{R}\,\mathrm{P}_{\mathrm{fin}}\,\mathbf{B}$ yields $\mathbb{A} \equiv \mathbb{B}^{[m']}(\beta)$. This completes the proof of (ii). By Corollary 2.12, α is invertible by $\mathrm{Cln}\,\mathbb{A}^{[m]} = \mathrm{Pol}(\mathbf{A}^m)$, thus (iii) holds.

"(ii)$\Longrightarrow$(i)". Let $\mathbb{B} \equiv \mathbb{A}^{[m]}(\alpha)$. That is, there is an algebra $\mathbb{B}' \simeq \mathbb{B}$ such that $\mathbb{B}' \equiv_{\mathrm{t}} \mathbb{A}^{[m]}(\alpha)$. Let $\mathbf{A}$ be any structure such that $\mathbb{A} \sim \mathbf{A}$ and define $\mathbf{B}' := \alpha(\mathbf{A}^m)$. By Corollaries 4.4 and 4.7, $\mathbb{B}' \sim \mathbf{B}'$. Renaming the elements of B according to the isomorphism $\mathbb{B}' \simeq \mathbb{B}$ gives an structure $\mathbf{B}$ isomorphic to $\mathbf{B}'$ such that $\mathbb{B} \sim \mathbf{B}$. By Remark 2.7, $\mathbf{B} \in \mathrm{R}\,\mathrm{P}_{\mathrm{fin}}\,\mathbf{A}$. Propositions 2.23 and 2.24 yield $\mathrm{Th}_{\mathrm{re}}\,\mathbf{A} \subseteq \mathrm{Th}_{\mathrm{re}}\,\mathbf{B}$. Hence, $\mathrm{Cln}\,\mathbf{A} \to \mathrm{Cln}\,\mathbf{B}$, so

$$\mathrm{Inv}\,\mathbb{A} \to \mathrm{Inv}\,\mathbb{B}.$$

Analogously, $\mathbb{A} \equiv \mathbb{B}^{[m']}(\beta)$ yields $\mathrm{Inv}\,\mathbb{B} \to \mathrm{Inv}\,\mathbb{A}$. By Proposition 3.12, $\mathrm{Inv}\,\mathbb{A} \leftrightarrow \mathrm{Inv}\,\mathbb{B}$. Theorem 4.9 completes the proof of (i).

"(iii)$\Longrightarrow$(i)". We just need to replace the argument showing $\mathrm{Inv}\,\mathbb{B} \to \mathrm{Inv}\,\mathbb{A}$ in the proof of "(ii)$\Longrightarrow$(i)". Since α is invertible by $\mathrm{Cln}\,\mathbb{A}^{[m]} = \mathrm{Pol}(\mathbf{A}^m)$, we have $\mathbf{A} \in \mathrm{R}\,\mathrm{P}_{\mathrm{fin}}\,\mathbf{B}$ (Corollary 2.12) and conclude as above $\mathrm{Inv}\,\mathbb{B} \to \mathrm{Inv}\,\mathbb{A}$. $\square$

4.3 Primal algebras

We give another application of Lemmas 4.3 and 4.6. Let $\mathrm{Eq}(A)$ denote the set of all equivalence relations on A. Let $\mathbb{A}$ be a finite algebra. Note that the congruences of $\mathbb{A}$ are exactly the relations in $\mathrm{Inv}\,\mathbb{A} \cap \mathrm{Eq}(A)$. A finite algebra $\mathbb{A}$ is called *congruence-primal*, or *hemi-primal*, if $\mathrm{Cln}\,\mathbb{A}$ is maximal among all clones of the form $\mathrm{Cln}\,\mathbb{A}'$, where $\mathbb{A}$ and $\mathbb{A}'$ have the same base set and possess the same congruences, i.e.,

$$\mathrm{Cln}\,\mathbb{A} = \mathrm{Pol}(\mathrm{Inv}\,\mathbb{A} \cap \mathrm{Eq}(A)).$$

We show that congruence-primality is preserved under formation of idempotent images and formation of matrix powers, hence, by Theorem 4.11, under categorical equivalence.

First, we observe that the property to be an equivalence relation can be expressed by relational equations (in Section 5.1.2 we give them explicitly). Thus, if $r \in \mathrm{Eq}(A)$ then, by Proposition 2.24, $r^{[m]} \in \mathrm{Eq}(A^m)$. Analogously, if α is an idempotent mapping in $\mathrm{End}\{r\}$ and $r \in \mathrm{Eq}(A)$ then, by Proposition 2.23, $\alpha(r) \in \mathrm{Eq}(\alpha(A))$. Together with Lemma 2.8 and Lemma 2.11 we obtain

$$r \in \mathrm{Eq}(A) \quad \text{iff} \quad r^{[m]} \in \mathrm{Eq}(A^m),$$

where $m \in \mathbb{N}_+$, and

$$r \in \mathrm{Eq}(A) \quad \text{iff} \quad \alpha(r) \in \mathrm{Eq}(\alpha(A)),$$

where α is an idempotent mapping in $\mathrm{End}\{r\}$ invertible by $\mathrm{Pol}\{r\}$. Hence for any $R \subseteq \mathrm{Rel}(A)$, $m \in \mathbb{N}_+$ and an idempotent $\alpha \in \mathrm{End}\,R$ invertible by $\mathrm{Pol}\,R$ we obtain

$$\begin{aligned}(R \cap \mathrm{Eq}(A))^{[m]} &= R^{[m]} \cap \mathrm{Eq}(A^m)\\ \alpha(R \cap \mathrm{Eq}(A)) &= \alpha(R) \cap \mathrm{Eq}(\alpha(A)).\end{aligned}$$

Let now $\mathbb{A}$ be a congruence-primal algebra. We check that the m-th matrix power $\mathbb{A}^{[m]}$ of it is congruence-primal. We calculate as follows.

$$\begin{aligned}
\operatorname{Cln}(\mathbb{A}^{[m]}) &= (\operatorname{Cln}\mathbb{A})^{[m]} \\
&= (\operatorname{Pol}(\operatorname{Inv}\mathbb{A} \cap \operatorname{Eq}(A)))^{[m]} \\
&= \operatorname{Pol}((\operatorname{Inv}\mathbb{A} \cap \operatorname{Eq}(A))^{[m]}) && \text{by 4.6} \\
&= \operatorname{Pol}((\operatorname{Inv}\mathbb{A})^{[m]} \cap \operatorname{Eq}(A^m)) && \text{see above} \\
&= \operatorname{Pol}((\operatorname{Inv}\operatorname{Cln}\mathbb{A})^{[m]} \cap \operatorname{Eq}(A^m)) && \text{by 1.6} \\
&= \operatorname{Pol}(\operatorname{Inv}((\operatorname{Cln}\mathbb{A})^{[m]}) \cap \operatorname{Eq}(A^m)) && \text{by 4.6} \\
&= \operatorname{Pol}(\operatorname{Inv}(\mathbb{A}^{[m]}) \cap \operatorname{Eq}(A^m))
\end{aligned}$$

An analogous calculation using Lemma 4.3 shows that congruence-primality is preserved under formation of idempotent images.

This fact has already been shown in [BB96], together with a characterization of the algebras categorically equivalent with a given congruence-primal algebra and the analogous considerations for subalgebra-primal and automorphism-primal algebras. But from the computation above, we can observe more: The only property of equivalence relations we use is that they are characterized by relational equations. Thus, the same conclusions can be drawn for instance for

- unary relations and subalgebra-primality (or semi-primality),
- graphs of functions and endomorphism-primality,
- graphs of bijective functions and automorphism-primality (or demi-primality),
- tolerances and tolerance-primality,

order relations and so on. In these cases, the required relational equations are easy to find, and are contained in the next chapter. The mentioned notions of primality are defined in an obvious way in analogy to the notion of congruence-primality.

Chapter 5

Special classes of structures

5.1 Structures with minimal clones

Definition 5.1. A clone $R \subseteq \mathrm{Rel}(A)$ is *minimal* if it is an atom in $\mathbf{L}_A$, i.e., $R \neq D(A)$ and for all clones R' with $D(A) \subseteq R' \subseteq R$ it holds $R' = D(A)$ or $R' = R$.

Clearly, a clone $R \neq D(A)$ is minimal if and only if for all $r' \in R \setminus D(A)$ we have $R = \mathrm{Cln}\{r'\}$. Let $\mathbf{A}$ and $\mathbf{B}$ be structures. Lemma 3.14 yields that if $\mathrm{Cln}\,\mathbf{A}$ is minimal and $\mathbf{B} \in \mathrm{Mod}\,\mathrm{Th}_{\mathrm{re}}\,\mathbf{A}$ then $\mathrm{Cln}\,\mathbf{B} = D(B)$ or $\mathrm{Cln}\,\mathbf{B}$ is minimal. Hence, it is possible to classify minimal clones by relational equations. By a result of I. G. Rosenberg [Ros70], the minimal clones over finite base sets can be classified into 6 types. In the following 6 subsections, we discuss for these types a characterization by relational equations. Further, in most cases, we will derive minimality directly from the characterizing relational equations. We call classes of structures of the form $\mathrm{Mod}\,\Sigma$, where Σ is a set of relational equations, *re-classes*. The emphasis of this section is on examples for re-classes and for the theory presented in the foregoing sections.

We denote variables by x, y and z (possibly using subscripts) preferring x for free variables and y for bounded variables. The proofs of the minimality are often technically involved and use similar arguments. We collect these arguments in the following method.

Method 5.2. By $\psi \in \varphi$ we denote that the atomic formula ψ is a subformula of the formula φ. Assume we are given a type consisting of one relation symbol r, and a set Σ of relational equations. We want to show that for any structure $\mathbf{A}$ satisfying Σ we have that $\mathrm{Cln}\,\mathbf{A}$ is minimal.

We have to show that for any $r' \in \mathrm{Cln}\,\mathbf{A}$ it holds $r' \in D(A)$ or $\mathsf{r}^{\mathbf{A}} \in \mathrm{Cln}\{r'\}$. Let $\varphi(x_0, \ldots, x_{m-1})$ be a primitive-positive formula such that $r' = \varphi^{\mathbf{A}}$. We can assume φ in the form

$$\varphi_{\approx} \wedge \varphi_{\mathsf{r}}$$

such that the following properties hold

- $\varphi_{\approx}$ is of the form $\bigwedge_k x_{1k} \approx x_{2k}$ with $x_{1k}, x_{2k} \in \{x_0, \ldots, x_{m-1}\}$,
- $\varphi_{\approx}$ is reflexive, i.e., $(x_i \approx x_i) \in \varphi_{\approx}$ for $i \in \{0, \ldots, m-1\}$,

- $\varphi_\approx$ is symmetric, i.e., $(x_i \approx x_j) \in \varphi_\approx$ if and only if $(x_j \approx x_i) \in \varphi_\approx$,
- $\varphi_\approx$ is transitive, i.e., if $(x_i \approx x_j) \in \varphi_\approx$ and $(x_j \approx x_k) \in \varphi_\approx$ then $(x_i \approx x_k) \in \varphi_\approx$,
- $\varphi_{\mathsf{r}} \in \Phi(\exists, \wedge)$ and φ_{r} does not contain two distinct variables x_i and x_j such that $(x_i \approx x_j) \in \varphi_\approx$.

It is easy to transform φ into an equivalent formula of this form.

Depending on Σ, we can assume φ_{r} to be of a more special form. Then a case study will yield

- $\varphi_{\mathsf{r}}^{\mathbf{A}} = A^{m'}$ for some $m' \leq m$, thus $\varphi^{\mathbf{A}} \in D(A)$, or
- $\varphi_{\mathsf{r}}^{\mathbf{A}} = \emptyset$, thus $\varphi^{\mathbf{A}} \in D(A)$, or
- $\mathsf{r}^{\mathbf{A}} \in \mathrm{Cln}\{r'\}$.

5.1.1 Directed orders

Let $\mathcal{R}$ consist of one binary relation symbol $\leq$. For sets X_1 and X_2 of variables we abbreviate by $X_1 \leq X_2$ the formula

$$\bigwedge_{x_1 \in X_1,\, x_2 \in X_2} x_1 \leq x_2.$$

Similarly, if $\mathbf{A}$ is an ordered set and $A_1, A_2 \subseteq A$ the notation $A_1 \leq A_2$ means $a_1 \leq a_2$ for all $a_1 \in A_1$ and $a_2 \in A_2$.

Definition 5.3. A structure $\mathbf{A} = (A, \leq^{\mathbf{A}})$ is a *directed order*, a *dor* for short, if $\leq^{\mathbf{A}}$ is an order relation and for any two elements $a, b \in A$ there is an upper bound and a lower bound for $\{a, b\}$.

A structure is a dor if and only if it satisfies the following relational equations.

$$x_0 \leq x_1 \leq x_0 \leftrightarrow x_0 \approx x_1 \tag{5.1}$$
$$(\exists y)\ x_0 \leq y \leq x_1 \leftrightarrow x_0 \leq x_1 \tag{5.2}$$
$$(\exists y)\ \{x_0, x_1\} \leq y \leftrightarrow \mathsf{t}(x_0, x_1) \tag{5.3}$$
$$(\exists y)\ y \leq \{x_0, x_1\} \leftrightarrow \mathsf{t}(x_0, x_1) \tag{5.4}$$

Equations (5.1) and (5.2) say that $\leq$ is an order relation and Equations (5.3) and (5.4) say that upper bounds and lower bounds exist as required. In the finite case, this means that the order is bounded, i.e., possessing a least element and a greatest element. An easy induction shows that from (5.1)–(5.4) it follows

$$(\exists y)\ \{x_0, \dots, x_{k-1}\} \leq y \leftrightarrow \mathsf{t}(x_0, \dots, x_{k-1}), \tag{5.5}$$
$$(\exists y)\ y \leq \{x_0, \dots, x_{k-1}\} \leftrightarrow \mathsf{t}(x_0, \dots, x_{k-1}). \tag{5.6}$$

Proposition 5.4. *Let* $\mathbf{A}$ *be a dor. Then* $\mathrm{Cln}\,\mathbf{A}$ *is minimal.*

Proof. We apply Method 5.2. By (5.1)–(5.4) we can assume φ_{r} in the following form.

- φ_{r} is asymmetric, i.e., there are no variables z_0, z_1 such that $(z_0 \leq z_1) \in \varphi_{\mathsf{r}}$ and $(z_1 \leq z_0) \in \varphi_{\mathsf{r}}$
- φ_{r} is transitive, i.e., if $(z_0 \leq z_1) \in \varphi_{\mathsf{r}}$ and $(z_1 \leq z_2) \in \varphi_{\mathsf{r}}$ then $(z_0 \leq z_2) \in \varphi_{\mathsf{r}}$

Case (1). There are no free variables x_i, x_j such that $(x_i \leq x_j) \in \varphi_{\mathsf{r}}$.

If there is any bounded variable in φ_{r} then there is a bounded variable y in φ_{r} such that for no z it holds $(z \leq y) \in \varphi_{\mathsf{r}}$ or for no z it holds $(y \leq z) \in \varphi_{\mathsf{r}}$. By (5.5), or (5.6) resp., deleting $\exists y$ and all atomic formulas containing y yields a formula with the same interpretation in $\mathbf{A}$. Iterating this, we delete all atomic formulas of φ_{r}. Hence, $\varphi_{\mathsf{r}}^{\mathbf{A}} = A^{m'}$ and $\varphi^{\mathbf{A}} \in D(A)$.

Case (2). There are free variables x_i, x_j such that $(x_i \leq x_j) \in \varphi_{\mathsf{r}}$.

We define φ' by prepending an $\exists$-quantifier to φ for all free variables of φ except x_i and x_j. Obviously, $\varphi'^{\mathbf{A}} \in \mathrm{Cln}\{r'\}$ and $\varphi'^{\mathbf{A}} \subseteq \leq^{\mathbf{A}}$. Given $a, b \in A$ with $a \leq^{\mathbf{A}} b$, we assign values to the variables z of φ' by

$$z \mapsto \begin{cases} a & \text{if } (z \approx x_i) \in \varphi_{\approx} \text{ or } (z \leq x_i) \in \varphi_{\mathsf{r}}, \\ b & \text{otherwise.} \end{cases}$$

Hence, $\langle a, b \rangle \in \varphi'^{\mathbf{A}}$, thus $\leq^{\mathbf{A}} = \varphi'^{\mathbf{A}} \in \mathrm{Cln}\{r'\}$. □

Next we consider re-classes contained in DOR, the class of all dor's. Clearly, the class $\mathbf{1}$ of one-element orders, defined by $\mathsf{t}(x_0, x_1) \leftrightarrow x_0 \approx x_1$, is the smallest such class. Another is the following.

Definition 5.5. A structure $\mathbf{A}$ is a *strongly directed order*, *sdor* for short, if it is a dor and obeys the relational equation

$$(\exists y)\ \{x_0, x_1\} \leq y \leq \{x_2, x_3\} \leftrightarrow \{x_0, x_1\} \leq \{x_2, x_3\}. \tag{5.7}$$

By SDOR we denote the class of all sdor's.

In the case of finite A this means that $\mathbf{A}$ is a lattice order. Example 5.7 below shows that this is not true for infinite A. An easy induction shows that all sdor's satisfy

$$(\exists y)\ \{x_0, \ldots, x_{k-1}\} \leq y \leq \{x'_0, \ldots, x'_{l-1}\} \leftrightarrow \{x_0, \ldots, x_{k-1}\} \leq \{x'_0, \ldots, x'_{l-1}\}. \tag{5.8}$$

In Figure 2.1 at page 16 a retraction from a distributive lattice order to a nonmodular lattice order is shown (the right arrows depict the retraction, the left arrows depict the coretraction). By Proposition 2.23, we obtain that we cannot express distributivity or modularity by relational equations in the type $\mathcal{R} = \{\leq\}$.

Furthermore, we have the following proposition.

Proposition 5.6. *For all re-classes K with $K \subseteq \mathrm{DOR}$ it holds $K = \mathbf{1}$ or $K \supseteq \mathrm{SDOR}$.*

Proof. For any primitive-positive formula φ we define a "normal form modulo dor" φ_{dor}. This means that φ_{dor} is a formula of the form $(\exists \bar{y}) \bigwedge_k z_{1k} \leq z_{2k}$ which is transitive, i.e.,

$$\text{if } (z_0 \leq z_1) \in \varphi_{\mathrm{dor}} \text{ and } (z_1 \leq z_2) \in \varphi_{\mathrm{dor}} \text{ then } (z_0 \leq z_2) \in \varphi_{\mathrm{dor}}, \tag{5.9}$$

and for any dor $\mathbf{A}$ we have $\varphi^{\mathbf{A}} = \varphi_{\mathrm{dor}}^{\mathbf{A}}$. Using (5.1)–(5.4) it is easy to transform φ into such a formula.

Analogously, φ_{sdor} is a formula of the form $\bigwedge_k z_{1k} \leq z_{2k}$ such that φ_{sdor} has the property (5.9) above and for any sdor $\mathbf{A}$ we have $\varphi^{\mathbf{A}} = \varphi_{\mathrm{sdor}}^{\mathbf{A}}$. By Equation (5.8), we obtain φ_{sdor} by deleting all $\exists$-quantifiers and all atomic formulas containing bounded variables from φ_{dor}.

Case (1). For all relational equations $\varphi_1 \leftrightarrow \varphi_2 \in \mathrm{Th}_{\mathrm{re}}\, K$ and for all free variables x_0, x_1 of φ_1 and φ_2 we have $(x_0 \leq x_1) \in \varphi_{1\mathrm{dor}}$ if and only if $(x_0 \leq x_1) \in \varphi_{2\mathrm{dor}}$.

Then $\varphi_{1\mathrm{sdor}}$ and $\varphi_{2\mathrm{sdor}}$ are equivalent, hence any sdor satisfies $\varphi_1 \leftrightarrow \varphi_2$, thus $K \supseteq \mathrm{SDOR}$.

Case (2). There is a relational equation $\varphi_1 \leftrightarrow \varphi_2 \in \mathrm{Th}_{\mathrm{re}}\, K$ and free variables x_0, x_1 of φ_1 and φ_2 such that $(x_0 \leq x_1) \in \varphi_{1\mathrm{dor}}$ and $(x_0 \leq x_1) \notin \varphi_{2\mathrm{dor}}$.

Let $\mathbf{A} \in K$ and $a, b \in A$ such that $a \neq b$. We can assume $a \leq b$. We assign values to the variables (bounded or free) in $\varphi_{1\mathrm{dor}}$ and $\varphi_{2\mathrm{dor}}$ by

$$z \mapsto \begin{cases} a & \text{if } (z \leq x_0) \in \varphi_{2\mathrm{dor}}, \\ b & \text{otherwise.} \end{cases}$$

This assignment, restricted to the free variables, defines a tuple belonging to $\varphi_{2\mathrm{dor}}^{\mathbf{A}}$ but not belonging to $\varphi_{1\mathrm{dor}}^{\mathbf{A}}$. This contradicts $K \subseteq \mathrm{DOR}$, thus K is the class of one-element orders. ☐

Example 5.7. The infinite structure

$$\big(\ \{N \subseteq \mathbb{N} \mid N \text{ finite or } |\mathbb{N} \setminus N| \leq 1\},\ \subseteq\ \big)$$

is an sdor but it is not a lattice order. Proposition 5.6 yields that for any nontrivial finite lattice order $\mathbf{A}$ (actually, for any nontrivial sdor $\mathbf{A}$) it holds $\mathrm{Mod}\,\mathrm{Th}_{\mathrm{re}}\, \mathbf{A} = \mathrm{SDOR}$. It is known that for any finite lattice order $\mathbf{A}$ the class $\mathrm{R\,P}\,\mathbf{A}$ consists of all complete lattice orders. Thus, this example shows that we can not omit the condition "$\mathbf{A}$ finite" in Theorems 2.27 and 2.28.

When we consider lattices *as algebras*, then a finite subset of a lattice can not necessarily be extended to a finite sublattice (but for distributive lattices it can). For the class SDOR the situation is nicer.

Lemma 5.8. *Let $\mathbf{A}$ be an sdor and let $B \subseteq A$ be finite. Then there exists a finite B' with $B \subseteq B' \subseteq A$ such that $\mathbf{B}'$, the substructure with base set B', is an sdor.*

Proof. For $A' \subseteq A$ we denote by $\mathrm{L}(A')$ the set of all lower bounds of A' and by $\mathrm{U}(A')$ the set of all upper bounds of A'.

We define a closure operator on B by

$$C \mapsto \overline{C} := \mathrm{L}(\mathrm{U}(C)) \cap B, \qquad C \subseteq B.$$

Let $C_1, \dots, C_n$ be all subsets of B with $C_i = \overline{C_i}$, arranged such that $C_i \supseteq C_j$ implies $i \leq j$.

Now we construct B' inductively as follows. We set $B_0 := \emptyset$ and $B_i := B_{i-1} \cup \{b_{C_i}\}$, for $i = 1, \dots, n$, where $b_{C_i} \in A$ has the following properties:

(i) $\mathrm{L}(\{b_{C_i}\}) \cap B = C_i$,

(ii) if $b \in B_{i-1}$ and $b \geq C_i$ then $b_{C_i} \leq b$,

(iii) if $C_i = \overline{\{b\}}$ for any $b \in B$ then $b_{C_i} = b$.

To see that this is possible we argue as follows. If $C_i = \overline{\{b\}}$ then it is possible to set b_{C_i} according to property (iii), since $b_1 \neq b_2$ implies $\overline{\{b_1\}} \neq \overline{\{b_2\}}$ for $b_1, b_2 \in B$. Obviously, this choice of b_{C_i} satisfies property (i) and (ii).

If C_i is not of the form $\overline{\{b\}}$ we construct b_{C_i} as follows. For any $b \in B \setminus C_i$ let $b' \in A$ be such that $b' \geq C_i$ and $b' \not\geq b$. Such a b' exists, since otherwise we would have $b \in \overline{C_i} = C_i$. Let $\tilde{B}$ be the set

$$\{b' \mid b \in B \setminus C_i\} \cup \{b \in B_{i-1} \mid b \geq C_i\}.$$

By construction, $C_i \leq \tilde{B}$. Equation (5.8) implies the existence of b_{C_i} with

$$C_i \leq b_{C_i} \leq \tilde{B}.$$

By construction, b_{C_i} satisfies property (i) and (ii).

Finally, we set $B' := B_n$. We claim that B' has the required property. Clearly, $\mathbf{B}'$ is an ordered set. By property (iii), $B \subseteq B'$. By property (ii), $\mathbf{B}'$ has the least element $b_{\overline{\emptyset}}$ and the largest element b_B. It remains to check Equation (5.7). Let $b_{C_i}, b_{C_j}, b_{C_k}, b_{C_l} \in B'$ such that $\{b_{C_i}, b_{C_j}\} \leq \{b_{C_k}, b_{C_l}\}$. Then

$$C_k = \mathrm{L}(\{b_{C_k}\}) \cap B \supseteq \mathrm{L}(\{b_{C_i}\}) \cap B = C_i,$$

and, similarly, $C_k \supseteq C_j$, $C_l \supseteq C_i$ and $C_l \supseteq C_j$. Let $C_m := \overline{C_i \cup C_j}$. Then

$$C_k, C_l \supseteq C_m \supseteq C_i, C_j,$$

thus $\{k, l\} \leq m \leq \{i, j\}$. By property (ii),

$$\{b_{C_i}, b_{C_j}\} \leq b_{C_m} \leq \{b_{C_k}, b_{C_l}\}.$$

□

Lemma 5.8 yields a generalization of Theorems 2.27 and 2.28 for sdor's.

Corollary 5.9. *Let* $\mathbf{A}$ *be an sdor. Let* K *be a class of structures such that* $\mathrm{Mod}\,\mathrm{Th}_{\mathrm{re}}\,K \supseteq \mathrm{SDOR}$. *Then* $\mathbf{A}$ *is the direct union of a directed family of structures from* $\mathrm{R}\,\mathrm{P}\,K$. *If, in addition,* K *is a finite class of finite structures then* $\mathbf{A}$ *is the direct union of a directed family of structures from* $\mathrm{R}\,\mathrm{P}_{\mathrm{fin}}\,K$.

We close this subsection with a remark about the lattice of re-classes contained in DOR. This looks like in Figure 5.1. S_{22} is the re-class generated by the structure with base set $\{0, 1, a_0, a_1, a_2, a_3\}$ and the ordering $0 \leq \{a_0, a_1\} \leq \{a_2, a_3\} \leq 1$. The part above S_{22} is more complicated, it is of infinite hight and of infinite width.

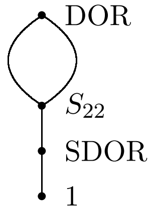

Figure 5.1: re-classes below DOR

In [DR81] the lattice of classes of ordered structures closed under products and retracts is partially described. This differs from our considerations in two points. We restrict ourself to directed orders and the concepts of re-classes and relational varieties coincide only under additional finiteness conditions (Theorems 2.27 and 2.28).

5.1.2 Equivalences

For the discussion of equivalences we choose the type $\mathcal{R}$ to contain one binary relation symbol $\sim$.

Definition 5.10. A structure $\mathbf{A} = (A, \sim^{\mathbf{A}})$ is an *equivalence* if $\sim^{\mathbf{A}}$ is an equivalence relation.

A structure is an equivalence if and only if it satisfies the following relational equations.

$$x_0 \sim x_0 \leftrightarrow \mathsf{t}(x_0) \tag{5.10}$$

$$x_0 \sim x_1 \leftrightarrow x_1 \sim x_0 \tag{5.11}$$

$$(\exists y)\ x_0 \sim y \sim x_1 \leftrightarrow x_0 \sim x_1 \tag{5.12}$$

These equations express reflexivity, symmetry and transitivity.

Proposition 5.11. *Let* $\mathbf{A}$ *be an equivalence. Then* $\operatorname{Cln} \mathbf{A}$ *is minimal.*

Proof. We apply Method 5.2. By (5.10)–(5.12) we can assume φ_{r} in the following form.

- φ_{r} does not contain $\exists$
- φ_{r} is symmetric, i.e., $(z_0 \sim z_1) \in \varphi_{\mathsf{r}}$ if and only if $(z_1 \sim z_0) \in \varphi_{\mathsf{r}}$
- φ_{r} is transitive, i.e., if $(z_0 \sim z_1) \in \varphi_{\mathsf{r}}$ and $(z_1 \sim z_2) \in \varphi_{\mathsf{r}}$ then $(z_0 \sim z_2) \in \varphi_{\mathsf{r}}$

If φ_{r} contain only atomic formulas of the form $x_i \sim x_i$, then $\varphi_{\mathsf{r}}^{\mathbf{A}} = A^{m'}$ and $\varphi^{\mathbf{A}} \in D(A)$. So let x_i, x_j be distinct free variables such that $(x_i \sim x_j) \in \varphi_{\mathsf{r}}$. We define φ' by prepending an $\exists$-quantifier to φ for all free variables of φ except x_i and x_j. Obviously, $\varphi'^{\mathbf{A}} \in \mathrm{Cln}\{r'\}$ and $\varphi'^{\mathbf{A}} \subseteq \sim^{\mathbf{A}}$. Given $a, b \in A$ with $a \sim^{\mathbf{A}} b$, we assign the value a to x_i and the value b to all other variables of φ'. Hence, $\langle a, b\rangle \in \varphi'^{\mathbf{A}}$, thus $\sim^{\mathbf{A}} = \varphi'^{\mathbf{A}} \in \mathrm{Cln}\{r'\}$. □

It is easy to check that the re-classes below the class of all equivalences are the class $\mathbf{1}$ of one-element equivalences, defined by $\mathsf{t}(x_0, x_1) \leftrightarrow x_0 \approx x_1$; and the class K_1 of all equivalences with one block, defined by $x_0 \sim x_1 \leftrightarrow \mathsf{t}(x_0, x_1)$; and the class K_2 of all discrete equivalences, defined by $x_0 \sim x_1 \leftrightarrow x_0 \approx x_1$.

5.1.3 Graphs of permutations

For the discussion of graphs of permutations we choose the type $\mathcal{R}$ to contain one binary relation symbol r. Let $\varphi(x, x')$ be a formula, possibly atomic. By $\varphi^l(x_0, x_1)$, $l \in \mathbb{N}_+$, we abbreviate the formula

$$(\exists y_0, \ldots, y_{l-2})\ \varphi(x_0, y_0) \wedge \varphi(y_0, y_1) \wedge \cdots \wedge \varphi(y_{l-3}, y_{l-2}) \wedge \varphi(y_{l-2}, x_1).$$

Definition 5.12. Let p be a prime. A structure $(A, \mathsf{r}^{\mathbf{A}})$ is a *p-permutation graph* if there is a fixed-point free permutation σ on A such that $\sigma^p = \mathrm{id}_A$ and $\mathsf{r}^{\mathbf{A}} = \{(a, \sigma(a)) \mid a \in A\}$.

A structure is a 2-permutation graph if and only if it satisfies the following relational equations.

$$\mathsf{r}(x_0, x_1) \leftrightarrow \mathsf{r}(x_1, x_0) \tag{5.13}$$
$$(\exists y)\ \mathsf{r}(x_0, y) \wedge \mathsf{r}(y, x_1) \leftrightarrow x_0 \approx x_1 \tag{5.14}$$
$$\mathsf{r}(x_0, x_0) \leftrightarrow \mathsf{f}(x_0) \tag{5.15}$$

Equations (5.13) and (5.14) say that $\mathsf{r}^{\mathbf{A}}$ is a graph of a permutation of order 2 and Equation (5.15) says that this permutation is fixed-point free.

A structure is a p-permutation graph, $p \geq 3$, if and only if it satisfies the following relational equations.

$$\mathsf{r}(x_0, x_1) \wedge \mathsf{r}(x_1, x_0) \leftrightarrow \mathsf{f}(x_0, x_1) \tag{5.16}$$
$$(\exists y)\ \mathsf{r}(x_0, y) \wedge \mathsf{r}(x_1, y) \leftrightarrow x_0 \approx x_1 \tag{5.17}$$
$$(\exists y)\ \mathsf{r}(y, x_0) \wedge \mathsf{r}(y, x_1) \leftrightarrow x_0 \approx x_1 \tag{5.18}$$
$$\mathsf{r}^p(x_0, x_1) \leftrightarrow x_0 \approx x_1 \tag{5.19}$$

Equations (5.16)–(5.18) say that $\mathsf{r}^{\mathbf{A}}$ is a graph of a fixed-point free permutation and Equation (5.19) says that this permutation is of order p.

Proposition 5.13. *Let* $\mathbf{A}$ *be a p-permutation graph. Then* $\mathrm{Cln}\,\mathbf{A}$ *is minimal.*

Proof. We omit the proof of the special case $p = 2$ and apply Method 5.2. We consider φ_{r} as a graph which vertices are the variables of φ_{r}, and there is an edge $z \to z'$ if $\mathsf{r}(z, z') \in \varphi_{\mathsf{r}}$. By (5.16)–(5.19) we can assume that the connected components of φ_{r} are chains, i.e., of the form

$$\underset{z_0}{\circ} \longrightarrow \underset{z_1}{\circ} \longrightarrow \quad \cdots \quad \longrightarrow \underset{z_l}{\circ}$$

of length $l < p$, or cycles, i.e. of the form

$$\underset{z_0}{\circ} \longrightarrow \underset{z_1}{\circ} \longrightarrow \quad \cdots \quad \longrightarrow \underset{z_{l-1}}{\circ} \longrightarrow \underset{z_0}{\circ}$$

of length $l < p$.

Case (1). φ_{r} contains a cycle of length $l < p$.

By (5.16) and (5.19), $\varphi_{\mathsf{r}}^{\mathbf{A}} = \emptyset$, so $\varphi^{\mathbf{A}} = \emptyset \in D(A)$.

Case (2). Case 1 does not hold and there are no free variables x_i, x_j such that φ_{r} contains a path from x_i to x_j.

By (5.17) and (5.18), $\varphi_{\mathsf{r}}^{\mathbf{A}} = A^{m'}$, so $\varphi^{\mathbf{A}} \in D(A)$.

Case (3). Case 1 does not hold and there are free variables x_i, x_j such that φ_{r} contains a path of length l from x_i to x_j.

We define φ' by prepending an $\exists$-quantifier to φ for all free variables of φ except x_i and x_j. Note that $\varphi'^{\mathbf{A}} = \mathsf{r}^{l\mathbf{A}}$. Let $l' \in \mathbb{N}_+$ be such that $ll' \equiv 1 \mod p$. We define φ'' by

$$\varphi'' := \varphi'^{l'}$$

By construction, $\varphi''^{\mathbf{A}} \in \mathrm{Cln}\{r'\}$. By (5.16)–(5.19), we have that $\varphi''^{\mathbf{A}} = \mathsf{r}^{\mathbf{A}}$. $\square$

5.1.4 Central structures

For the discussion of central structures we fix an arbitrary $m \in \mathbb{N}_+$ and choose the type $\mathcal{R}$ to contain one m-ary relation symbol r.

Definition 5.14. Let A be a set, let $m \in \mathbb{N}_+$ and let $r \in \mathrm{Rel}^{(m)}(A)$. The relation r is *totally symmetric* if for any permutation σ of $\{0, \dots, m-1\}$ it holds $r(a_0, \dots, a_{m-1})$ if and only if $r(a_{\sigma(0)}, \dots, a_{\sigma(m-1)})$. The relation r is *totally reflexive* if for any elements $a_0, \dots, a_{m-1}$ which are not pairwise distinct it holds $r(a_0, \dots, a_{m-1})$. Let now A be finite. The relation r is *central* if it is totally symmetric, totally reflexive and there is a $c \in A$ such that $c \in \{a_0, \dots, a_{m-1}\}$ implies $r(a_0, \dots, a_{m-1})$. We call a finite structure $(A, \mathsf{r}^{\mathbf{A}})$ central if $\mathsf{r}^{\mathbf{A}}$ is central.

A finite structure is central if and only if it satisfies the following relational equations

$$\mathsf{r}(x_0, \dots, x_{m-1}) \leftrightarrow \mathsf{r}(x_1, x_0, x_2, \dots, x_{m-1}), \tag{5.20}$$

$$\mathsf{r}(x_0, \dots, x_{m-1}) \leftrightarrow \mathsf{r}(x_1, \dots, x_{m-1}, x_0), \tag{5.21}$$

$$\mathsf{r}(x_0, x_0, \dots, x_{m-2}) \leftrightarrow \mathsf{t}(x_0, \dots, x_{m-2}), \tag{5.22}$$

and the infinite sequence of relational equations

$$\begin{aligned}(\exists y)\ r(y, x_0, \dots, x_{m-2}) \wedge \cdots \wedge r(y, x_{(n-1)(m-1)}, \dots, x_{n(m-1)-1}) \\ \leftrightarrow \mathsf{t}(x_0, \dots, x_{n(m-1)-1}), \qquad \text{for all } n \in \mathbb{N}_+.\end{aligned} \tag{5.23}$$

If we dropped the restriction on A to be finite in Definition 5.14, it would not be possible to characterize central structures by relational equations. Therefore, we define a structure, finite or infinite, to be *central* if it obeys Equations (5.20)–(5.23).

The following example shows that we cannot replace Equations (5.20)–(5.23) by a finite set of relational equations. Let $A := \{0, \ldots, l-1, c_0, \ldots, c_{l-1}\}$ and define a binary relation $r^{\mathbf{A}}$ on A by

$$r^{\mathbf{A}} := \{(c_i, c_j)\} \cup \{(i, c_j) \mid i \neq j\} \cup \{(i, i)\},$$

with $i, j \in \{0, \ldots, l-1\}$. Then the structure $(A, r^{\mathbf{A}})$ satisfies Equations (5.20)–(5.22) and all relational equations of (5.23) with $n < l$, but it does not satisfy any relational equation of (5.23) with $n \geq l$.

Proposition 5.15. *Let* $\mathbf{A}$ *be central. Then* $\operatorname{Cln} \mathbf{A}$ *is minimal.*

Proof. We apply Method 5.2. By (5.20)–(5.23) we can assume that φ_{r} does not contain $\exists$. If φ_{r} contains no atomic formula $\mathsf{r}(x_0, \ldots, x_{m-1})$ with pairwise distinct variables $x_0, \ldots, x_{m-1}$, then, by (5.22), $\varphi_{\mathsf{r}}^{\mathbf{A}} = A^{m'}$ and $\varphi^{\mathbf{A}} \in D(A)$. So let $r(x_0, \ldots, x_{m-1}) \in \varphi_{\mathsf{r}}$ with pairwise distinct variables $x_0, \ldots, x_{m-1}$. We define φ' by prepending an $\exists$-quantifier to φ for all free variables of φ except $x_0, \ldots, x_{m-1}$. Clearly, $\varphi'^{\mathbf{A}} \in \operatorname{Cln}\{r'\}$. By (5.23), $\varphi'^{\mathbf{A}} = \mathsf{r}^{\mathbf{A}}$. ☐

5.1.5 Quasilinear structures

For the discussion of quasilinear structures we fix an arbitrary prime number p and choose the type $\mathcal{R}$ to contain the relation symbols $\mathsf{r}_2, \ldots, \mathsf{r}_p$ with $\operatorname{ar}(\mathsf{r}_i) = 2i$.

Definition 5.16. A structure $(A, \mathsf{r}_2^{\mathbf{A}}, \ldots, \mathsf{r}_p^{\mathbf{A}})$ is *quasilinear* if there is an abelian group $(A, +, -, 0)$ with base set A satisfying $px := \underbrace{x + x + \cdots + x}_{p\text{-times}} = 0$ such that

$$\mathsf{r}_i = \{\langle a_0, \ldots, a_{2i-1}\rangle \mid a_0 + \cdots + a_{i-1} = a_i + \cdots + a_{2i-1}\}, \qquad i \in \{2, \ldots, p\}.$$

For a quasilinear structure $\mathbf{A}$ we have that $\operatorname{Cln} \mathbf{A} = \operatorname{Cln}\{\mathsf{r}_2^{\mathbf{A}}\}$. Thus, in our discussion, we could omit all symbols except r_2 from $\mathcal{R}$ and define structures accordingly. However, the choice made above enables an easier presentation of relational equations characterizing quasilinear structures.

Proposition 5.17. *A structure is quasilinear if and only if it satisfies the following relational equations. (The idea of this equations becomes clear in the proof below.)*

$$\mathsf{r}_2(x_0, x_1, x_2, x_3) \leftrightarrow \mathsf{r}_2(x_1, x_0, x_2, x_3) \tag{5.24}$$
$$\mathsf{r}_2(x_0, x_1, x_2, x_3) \leftrightarrow \mathsf{r}_2(x_2, x_3, x_0, x_1) \tag{5.25}$$
$$(\exists y)\mathsf{r}_2(x_0, x_1, x_2, y) \leftrightarrow \mathsf{t}(x_0, x_1, x_2) \tag{5.26}$$
$$\mathsf{r}_2(x_0, x_1, x_2, x_3) \wedge \mathsf{r}_2(x_0, x_1, x_2, x_3') \rightarrow x_3 \approx x_3' \tag{5.27}$$
$$\mathsf{r}_2(x_0, x_1, x_0, x_2) \leftrightarrow x_1 \approx x_2 \tag{5.28}$$
$$\mathsf{r}_3(x_0, x_1, x_2, x_3, x_4, x_5) \leftrightarrow \mathsf{r}_3(x_0, x_2, x_1, x_3, x_4, x_5) \tag{5.29}$$
$$\mathsf{r}_p(\underbrace{x_0, \dots, x_0}_{p\text{-times}}, \underbrace{x_1, \dots, x_1}_{p\text{-times}}) \leftrightarrow \mathsf{t}(x_0, x_1) \tag{5.30}$$
$$\mathsf{r}_2(x_0, x_1, x_2, x_3) \wedge \mathsf{r}_2(x_0', x_1', x_2, x_3) \rightarrow \mathsf{r}_2(x_0, x_1, x_0', x_1') \tag{5.31}$$

and the sequence of relational equations

$$\begin{aligned} &\mathsf{r}_i(x_0, \dots, x_{2i-1}) \\ &\quad \leftrightarrow (\exists y)\mathsf{r}_{i-1}(x_0, \dots, x_{i-2}, x_i, \dots, x_{2i-3}, y) \wedge \mathsf{r}_2(y, x_{i-1}, x_{2i-2}, x_{2i-1}) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{for } i = 3, \dots, p \end{aligned} \tag{5.32}$$

Proof. It is an immediate consequence of Definition 5.16, that a quasilinear structure satisfies Equations (5.24)–(5.32).

Let $\mathbf{A}$ be a structure satisfying Equations (5.24)–(5.32). We construct a group $(A, +, -, 0)$ witnessing that $\mathbf{A}$ is quasilinear. We fix any element of A to be 0 and define $+$ by

$$a_0 + a_1 = a \quad \text{iff} \quad \mathsf{r}_2^{\mathbf{A}}(a_0, a_1, 0, a).$$

By (5.26) and (5.27), this defines a binary operation, which is commutative by (5.24). Now (5.32) implies that

$$(a_0 + a_1) + a_2 = a \quad \text{iff} \quad \mathsf{r}_3^{\mathbf{A}}(a_0, a_1, a_2, 0, 0, a),$$

thus, by (5.29), $+$ is associative. The element 0 is neutral since it holds $\mathsf{r}_2^{\mathbf{A}}(0, a, 0, a)$ for all $a \in A$ (Equation (5.28)). Inverses exist since for any $a \in A$ there is an $a' \in A$ such that $\mathsf{r}_2^{\mathbf{A}}(a, a', 0, 0)$ (Equations (5.25) and (5.26)). Finally, (5.32) implies that

$$a_0 + \dots + a_{p-1} = 0 \quad \text{iff} \quad \mathsf{r}_p^{\mathbf{A}}(a_0, \dots, a_{p-1}, 0, \dots, 0),$$

thus, by (5.30), it hold $pa = 0$ for all $a \in A$.

It remains to show that $(A, +, -, 0)$ defines $\mathbf{A}$ as in Definition 5.16. Let $\mathbf{A}'$ be the structure defined by $(A, +, -, 0)$. It holds

$$\mathsf{r}_2^{\mathbf{A}'}(a_0, a_1, a_2, a_3) \quad \text{iff} \quad a_0 + a_1 = a_2 + a_3 \quad \text{iff} \quad \mathsf{r}_2^{\mathbf{A}}(a_0, a_1, 0, a_2 + a_3).$$

Since $\mathsf{r}_2^{\mathbf{A}}(a_2, a_3, 0, a_2 + a_3)$ and by (5.31) this is equivalent to

$$\mathsf{r}_2^{\mathbf{A}}(a_0, a_1, a_2, a_3).$$

Hence, $\mathsf{r}_2^{\mathbf{A}} = \mathsf{r}_2^{\mathbf{A}'}$. Because both $\mathbf{A}$ and $\mathbf{A}'$ satisfy Equation (5.32) and the interpretation of r_2 and Equation (5.32) uniquely determine the interpretations of $\mathsf{r}_3, \dots, \mathsf{r}_p$, we have $\mathbf{A} = \mathbf{A}'$. $\square$

Since quasilinear structures are known to possess minimal clones, Proposition 5.17 shows that any structure satisfying Equations (5.24)–(5.32) do so. It is messy to derive this fact directly from the equations. We give here a rough and informal description how to show that the clone of a quasilinear structure is minimal. In Proposition 5.33 we describe, in the special case $p = 2$, how this arguments interplay with relational equations. Let $\mathbf{A}$ be a quasilinear structure and φ be a primitive-positive formula. Then we can conclude as follows.

- $\varphi^{\mathbf{A}}$ is the set of solutions of a system of homogeneous linear equations φ_j, $j = 1, \ldots, l$, of the form
$$\sum\nolimits_{i=0}^{m-1} n_{j,i} x_i \approx 0 \qquad \text{with } \sum\nolimits_{i=0}^{m-1} n_{j,i} = 0,$$
where the coefficients $n_{j,i}$ belong to the p-element field. We identify φ and this system.
- Let n_{j,i_j} be the first nonzero coefficient in φ_j. The system φ is equivalent to one in canonical form, that is, we can assume the sequence $i_1, \ldots, i_l$ strictly increasing.
- All linear equations with at most one nonzero coefficient, are satisfied by all $x_0, \ldots, x_{m-1}$. All linear equations with exactly two nonzero coefficients, say $n_{j,i}$ and $n_{j,i'}$, are equivalent to $x_i \approx x_{i'}$. According to Method 5.2 we can assume that φ does not contain linear equations with less than three nonzero coefficients.
- If φ contains no linear equation with at least three nonzero coefficients, then $\varphi^{\mathbf{A}} \in D(A)$ and we are done. So we can assume that there are only linear equations with at least three nonzero coefficients. Let $\varphi' := (\exists x_0, \ldots, x_{i_l - 1})\ \varphi$. Then ${\varphi'}^{\mathbf{A}} = {\varphi_l}^{\mathbf{A}}$.
- We identify linear equations and the set of their solutions. By primitive-positive definitions, we derive successively from ${\varphi'}^{\mathbf{A}}$
  - a linear equation with at least p nonzero coefficients, hence two coefficients must be equal
  - a linear equation with two coefficients equal to 1,
  - a linear equation of the form $x_1 + x_2 + (p-2)x_3$,

  and finally $\mathsf{r}_2^{\mathbf{A}}$.

5.1.6 Universal structures

For the discussion of universal structures we fix an arbitrary $m \in \mathbb{N}_+$ with $m \geq 3$ and choose the type $\mathcal{R}$ to contain one m-ary relation symbol r.

Definition 5.18. The structure $\mathbf{E}_m$ is defined to have the base set $\{0, \ldots, m-1\}$ and the base relation
$$\mathsf{r}^{\mathbf{E}_m} := \{\langle a_0, \ldots, a_{m-1}\rangle \mid |\{a_0, \ldots, a_{m-1}\}| < m\},$$

i.e., the set of all m-tuples which components are not pairwise distinct. A structure $\mathbf{A}$ is *universal* if there is an $l \in \mathbb{N}_+$ and a strong homomorphism $\alpha\colon \mathbf{A} \to {\mathbf{E}_m}^l$.

Since the following considerations are similar for different m, we restrict ourself to the case $m = 3$ and fix $\mathcal{R}$ to contain one 3-ary symbol r. Although I failed to find a set of relational equations characterizing universal structures, we obtain a pretty good insight into this class of structures. This is based on the following lemma.

Lemma 5.19. *Let φ be any relational equation such that $\mathbf{E}_3 \models \varphi$. Then there is a $\varphi' \in \Sigma(\exists, \wedge, \mathsf{f}, \mathsf{t})$, i.e., a relational equation not containing $\approx$, such that φ and φ' are equivalent.*

Proof. We assume φ in the form $\varphi_1(x_0, \dots, x_{m-1}) \to \varphi_2(x_0, \dots, x_{m-1})$. As noted below Definition 1.4 we can assume that atomic subformulas containing $\approx$ involve only variables $x_0, \dots, x_{m-1}$. Without loss of generality we consider only occurrences of the form $x_0 \approx x_1$.

If $(x_0 \approx x_1) \in \varphi_1$, then we can transform φ equivalently as follows

$$\begin{aligned}
\varphi_1(x_0, \dots, x_{m-1}) &\to \varphi_2(x_0, \dots, x_{m-1}),\\
x_0 \approx x_1 \wedge \varphi_1(x_0, \dots, x_{m-1}) &\to \varphi_2(x_0, \dots, x_{m-1}),\\
x_0 \approx x_1 \wedge \varphi_1(x_0, \dots, x_{m-1}) &\to x_0 \approx x_1 \wedge \varphi_2(x_0, \dots, x_{m-1}),
\end{aligned}$$

and by renaming all occurrences of x_0 in φ_1 and φ_2 by x_1,

$$\begin{aligned}
x_0 \approx x_1 \wedge \varphi_1'(x_1, \dots, x_{m-1}) &\to x_0 \approx x_1 \wedge \varphi_2'(x_1, \dots, x_{m-1}),\\
\varphi_1'(x_1, \dots, x_{m-1}) &\to \varphi_2'(x_1, \dots, x_{m-1}).
\end{aligned}$$

Iterating this, we remove all occurrences of $\approx$ from φ_1.

So, we can assume that $\approx$ does not occur in φ_1. We are done, if we can show that $\approx$ does not occur in φ_2. Assume, to the contrary, that $(x_0 \approx x_1) \in \varphi_2$. Since

$$\mathbf{E}_3 \models \varphi_1(x_0, \dots, x_{m-1}) \to x_0 \approx x_1 \wedge \varphi_2(x_0, \dots, x_{m-1}),$$

we have

$$\mathbf{E}_3 \models \varphi_1(x_0, \dots, x_{m-1}) \to x_0 \approx x_1.$$

Now we assign 0 to x_0 and 1 to all other variables (free or bounded). Since $\{a_0, a_1, a_2\} \subseteq \{0, 1\}$ implies $\langle a_0, a_1, a_2 \rangle \in \mathsf{r}^{\mathbf{E}_3}$ this defines a tuple contained in $\varphi_1^{\mathbf{E}_3}$. But $0 \neq 1$, a contradiction. □

Corollary 5.20. *For any universal structure $\mathbf{A}$ it holds $\mathrm{Th}_{\mathrm{re}}\, \mathbf{A} = \mathrm{Th}_{\mathrm{re}}\, \mathbf{E}_3$.*

Proof. Let $\alpha\colon \mathbf{A} \to {\mathbf{E}_3}^l$ be a strong homomorphism. By Lemma 2.8 and Propositions 2.23 and 2.24 we have $\mathrm{Th}_{\mathrm{re}}\, {\mathbf{E}_3}^l = \mathrm{Th}_{\mathrm{re}}\, \mathbf{E}_3$ for any $l \in \mathbb{N}_+$. By Proposition 2.25(i) and Lemma 5.19 we have $\mathrm{Th}_{\mathrm{re}}\, \mathbf{A} = \mathrm{Th}_{\mathrm{re}}\, {\mathbf{E}_3}^l$. □

The problem to determine relational equations which characterize universal structures reduces to the following problem.

Problem 5.21. Find a set Σ of relational equations such that $\Sigma \models \mathrm{Th}_{\mathrm{re}}\, \mathbf{E}_3$. Can Σ be chosen to be finite?

5.2 Rigid structures

If a finite algebra $\mathbb{A}$ satisfies $\mathrm{Cln}\,\mathbb{A} = \mathrm{Func}(A)$, i.e., it is primal, then the variety generated by $\mathbb{A}$ can be described in a special form. In fact, $\mathrm{V}(\mathbb{A})$ consists of all algebras isomorphic to a Boolean power of $\mathbb{A}$ [BS81, §IV.7]. Does a similar result hold for relational structures?

Definition 5.22. A finite relational structure $\mathbf{A}$ is *rigid* if $\mathrm{Pol}\,\mathbf{A} = P(A)$.

By Theorem 1.6, we have that for a finite structure $\mathbf{A}$ it holds $\mathrm{Cln}\,\mathbf{A} = \mathrm{Rel}(A)$ if and only if $\mathrm{Pol}\,\mathbf{A} = P(A)$. That is why, we could say 'relationally primal' instead of 'rigid'. In the literature, the term rigid is often used in a weaker sense, namely, to express that $\mathrm{Aut}\,\mathbf{A} = \{\mathrm{id}\}$ or to express that $\mathrm{End}\,\mathbf{A} = \{\mathrm{id}\}$. General approaches to notions of rigidity of relational structures can be found in [Ros73, GP94, Fea95, GP96]. In the following lemma, we weak rigidity in an other way: We assume only $\mathrm{Cln}\,\mathbf{A} \supseteq \mathrm{Inv}\,\mathrm{Aut}\,\mathbf{A}$ (instead of $\mathrm{Cln}\,\mathbf{A} = \mathrm{Rel}(A)$). By Theorem 1.6, we have that $\mathrm{Cln}\,\mathbf{A} \supseteq \mathrm{Inv}\,\mathrm{Aut}\,\mathbf{A}$ holds if and only if the clone $\mathrm{Pol}\,\mathbf{A}$ is generated by unary, bijective functions.

Lemma 5.23. *Let $\mathbf{A}$ be a finite relational structure such that*

$$\mathrm{Cln}\,\mathbf{A} \supseteq \mathrm{Inv}\,\mathrm{Aut}\,\mathbf{A}.$$

Then the set $\mathrm{Th}_{\mathrm{re}}\,\mathbf{A}$ is equivalent to the set $\mathrm{Th}_{\mathrm{Horn}}\,\mathbf{A}$ of all Horn-sentences satisfied by $\mathbf{A}$.

Proof. Since, by Lemma 2.21, every relational equation is equivalent to a Horn-sentence we have that $\mathrm{Th}_{\mathrm{Horn}}\,\mathbf{A} \models \mathrm{Th}_{\mathrm{re}}\,\mathbf{A}$.

To prove the converse, for any Horn-sentence $\varphi^* \in \mathrm{Th}_{\mathrm{Horn}}\,\mathbf{A}$, we have to find a set $\Phi \subseteq \mathrm{Th}_{\mathrm{re}}\,\mathbf{A}$ such that $\Phi \models \varphi^*$.

By Theorem 1.6, the assumption $\mathrm{Cln}\,\mathbf{A} \supseteq \mathrm{Inv}\,\mathrm{Aut}\,\mathbf{A}$ is equivalent to that for any first-order formula φ there is a primitive-positive formula φ' such that $\varphi^{\mathbf{A}} = \varphi'^{\mathbf{A}}$.

If φ^* is a universal Horn-sentence then, by Lemma 2.21, it is equivalent to a relational equation, and we are done. So, let us assume φ^* in the form

$$(\forall \bar{x})(\exists y)(Q\bar{z})\ \bigwedge\nolimits_{j\in J}(\varphi_j \to \psi_j),$$

where $(\forall \bar{x})$ is possibly omitted, $(Q\bar{z})$ is any sequence of quantifiers, the φ_j are conjunctions of atomic formulas, and the ψ_j are atomic formulas. Let $\varphi(\bar{x}, y)$ be the formula $(Q\bar{z})\ \bigwedge_{j\in J}(\varphi_j \to \psi_j)$. As noted above, there is a primitive-positive formula φ' such that $\varphi^{\mathbf{A}} = \varphi'^{\mathbf{A}}$. Let φ' be of the form $(\exists \bar{w})\ \varphi''$, where φ'' is a conjunction of atomic formulas. Thus, $\mathbf{A}$ satisfies $\varphi^+ := (\forall \bar{x})(\exists y)\ \varphi'(\bar{x}, y)$. We note that φ^+ can be equivalently written in the form $(\exists y)\ \varphi'(\bar{x}, y) \leftrightarrow \mathsf{t}(\bar{x})$, thus it is a relational equation.

Moreover, $\mathbf{A}$ satisfies the sentence

$$(\forall \bar{x})(\forall y)\ \varphi' \ \to\ \Big((Q\bar{z})\ \bigwedge\nolimits_j(\varphi_j \to \psi_j)\Big),$$

which is equivalent to

$$\varphi^{++} := (\forall \bar{x})(\forall y)(\forall \bar{w})(Q\bar{z}) \ \bigwedge_j (\varphi'' \wedge \varphi_j \to \psi_j).$$

Note that φ^{++} is a Horn-sentence with strictly fewer $\exists$-quantifiers than φ^*, and that $\{\varphi^+, \varphi^{++}\} \models \varphi^*$. Iterating this argument we end up with a set of relational equations satisfying the claim. $\square$

By a result of J. Keisler [Kei65], we have that for any class K of structures and any structure $\mathbf{B}$ it holds $\mathbf{B} \in \operatorname{Mod}\operatorname{Th}_{\mathrm{Horn}} K$ if and only if there is an ultrapower of $\mathbf{B}$ isomorphic to a reduced product of structures from K. This yields the following corollary.

Corollary 5.24. *Let* $\mathbf{A}$ *be a structure satisfying the assumption of Lemma 5.23 and let* $\mathbf{B}$ *be any structure. Then* $\mathbf{B} \in \operatorname{Mod}\operatorname{Th}_{\mathrm{re}} \mathbf{A}$ *if and only if there is an ultrapower of* $\mathbf{B}$ *isomorphic to a reduced power of* $\mathbf{A}$*.*

Let $\mathbf{A}$ be a structure satisfying the assumption of Lemma 5.23. Then for any finite $\mathbf{B} \in \operatorname{Mod}\operatorname{Th}_{\mathrm{re}} \mathbf{A}$ we have $\mathbf{B} \in \mathrm{P}_{\mathrm{fin}} \mathbf{A}$. By Theorem 2.28, this is equivalent to $\mathrm{R}\,\mathrm{P}_{\mathrm{fin}} \mathbf{A} = \mathrm{P}_{\mathrm{fin}} \mathbf{A}$. If $\operatorname{End}\mathbf{A} \neq \operatorname{Aut}\mathbf{A}$ this is not true. Indeed, for any $\alpha \in \operatorname{End}\mathbf{A} \setminus \operatorname{Aut}\mathbf{A}$ any idempotent iterate of α defines a retract $\mathbf{B}$ of $\mathbf{A}$ with $|B| < |A|$, hence $\mathbf{B} \in \mathrm{R}\,\mathbf{A}$ but $\mathbf{B} \notin \mathrm{P}\,\mathbf{A}$. The next Proposition shows that we can simplify $\mathrm{R}\,\mathrm{P}\,\mathbf{A}$ even in this cases.

Proposition 5.25. *Let* $\mathbf{A}$ *be a finite relational structure such that*

$$\operatorname{Cln}\mathbf{A} \supseteq \operatorname{Inv}\operatorname{End}\mathbf{A}.$$

Then $\mathrm{R}\,\mathrm{P}\,\mathbf{A} = \mathrm{I}\,\mathrm{P}\,\mathrm{R}\,\mathbf{A}$*.*

Proof. For all structures $\mathbf{A}$, it holds $\mathrm{I}\,\mathrm{P}\,\mathrm{R}\,\mathbf{A} \subseteq \mathrm{R}\,\mathrm{P}\,\mathbf{A}$ (Lemma 2.9). Conversely, let $\mathbf{B} \in \mathrm{R}\,\mathrm{P}\,\mathbf{A}$, i.e., there is a power $\mathbf{C} = \mathbf{A}^I$ and a retraction $\alpha\colon \mathbf{C} \to \mathbf{B}$. By Remark 2.7, we can assume that $\alpha\colon \mathbf{C} \to \mathbf{C}$ is an idempotent endomorphism of $\mathbf{C}$ and $\mathbf{B} = \alpha\mathbf{C}$. We have to show that $\mathbf{B}$ is isomorphic to a structure in $\mathrm{P}\,\mathrm{R}\,\mathbf{A}$.

By Theorem 1.6 we obtain

$$\operatorname{Pol}\mathbf{A} = \operatorname{Pol}\operatorname{Cln}\mathbf{A} = \operatorname{Pol}\operatorname{Inv}\operatorname{End}\mathbf{A} = \operatorname{Cln}\operatorname{End}\mathbf{A},$$

that is, $\operatorname{Pol}\mathbf{A}$ consists of essentially unary functions.

For $j \in I$, let $\tilde{\alpha}_j\colon \mathbf{C} \to \mathbf{A}$ be the j-th component map of α, i.e, the map given by $c \mapsto \alpha(c)(j)$, $c \in C$.

Claim (*). $\tilde{\alpha}_j$ is essentially unary, i.e., there is an $i_j \in I$ and an $\alpha_j \in \operatorname{End}\mathbf{A}$ such that $\tilde{\alpha}_j(c) = \alpha_j(c(i_j))$ for all $c \in C$.

Proof of Claim (*). It is easy to check (see, e.g. [PK79]) that $\operatorname{Pol}\mathbf{A}$ consists of essentially unary functions if and only if the relation

$$r := \{\langle a_0, a_1, a_2, a_3\rangle \mid a_0 = a_1 \text{ or } a_2 = a_3\}$$

belongs to $\operatorname{Cln}\mathbf{A}$.

By Lemmas 2.13 and 2.16, it follows that for $c_0, c_1, c_2, c_3 \in C$ with

$$\langle c_0(i), c_1(i), c_2(i), c_3(i)\rangle \in r \qquad \text{for all } i \in I$$

we have

$$\langle \tilde{\alpha}_j(c_0), \tilde{\alpha}_j(c_1), \tilde{\alpha}_j(c_2), \tilde{\alpha}_j(c_3)\rangle \in r.$$

This implies that $\tilde{\alpha}_j$ is essentially unary. (If I is finite, we can argue shorter that $\tilde{\alpha}_j$ is a polymorphism of $\mathbf{A}$ and thus essentially unary.) Since α is a homomorphism, so is α_j. This completes the proof of the claim.

To make the representation of α unique, we agree to set $i_j = j$ if α_j is constant.

For $i \in I$, let J_i be the set of all j such that $\alpha(c)(j) = \alpha_j(c(i))$. Let I' be the set of all i where $J_i \neq \emptyset$.

Claim (**). For all $i \in I'$ it holds $i \in J_i$.

Proof of Claim(**). Let $\alpha(c)(i) = \alpha_i(c(i'))$ and let $j \in J_i$, i.e., $\alpha(c)(j) = \alpha_j(c(i))$. If α_j is constant then $i = j$ by the agreement above and we are done. Hence we can assume that α_j is not constant and choose $a, a' \in A$ such that $\alpha_j(a) \neq \alpha_j(\alpha_i(a'))$. Assume, to the contrary, $i \neq i'$. Let c be any element in C with $c(i) = a$ and $c(i') = a'$. Because α is idempotent we have $\alpha(\alpha(c)) = \alpha(c)$. But $\alpha(\alpha(c))(j) = \alpha_j(\alpha_i(a'))$ and $\alpha(c)(j) = \alpha_j(a)$, a contradiction. Thus $i = i'$ and the claim is verified.

Now, we decompose $\mathbf{B}$ and α into "blocks" according to the J_i. For $i \in I'$, let $\alpha_{J_i}\colon \mathbf{A} \to \mathbf{A}^{J_i}$ be the homomorphism defined by $\alpha_{J_i}(a)(j) := \alpha_j(a)$. Let $\mathbf{B}_i := \alpha_{J_i}\mathbf{A}$. By construction, $\mathbf{B}$ is isomorphic to $\prod_{i\in I'} \mathbf{B}_i$.

It remains to show that each $\mathbf{B}_i$ is isomorphic to a retract of $\mathbf{A}$. To see this, let $\pi_i\colon \mathbf{A}^{J_i} \to \mathbf{A}$ be the projection map to the i-th component, and let $\bar{\alpha}_{J_i} := \alpha_{J_i}\pi_i\colon A^{J_i} \to A^{J_i}$. Since α is an idempotent, so is $\bar{\alpha}_{J_i}$. That is,

$$\alpha_{J_i}\pi_i\alpha_{J_i}\pi_i = \bar{\alpha}_{J_i}\bar{\alpha}_{J_i} = \bar{\alpha}_{J_i} = \alpha_{J_i}\pi_i$$

and, since π_i is surjective, this implies $\alpha_{J_i}\pi_i\alpha_{J_i} = \alpha_{J_i}$. Hence, $\alpha_{J_i}\pi_i\restriction_{B_i} = \mathrm{id}_{B_i}$. By Lemma 2.8, π_i is a homomorphism, thus $(\alpha_{J_i}, \pi_i\restriction_{B_i})\colon \mathbf{A} \to \mathbf{B}_i$ is a retraction. $\square$

Under the stronger assumption $\mathrm{Cln}\,\mathbf{A} \supseteq \mathrm{Inv}\,\mathrm{Aut}\,\mathbf{A}$ ($\mathbf{A}$ finite), we have $\mathrm{R}\,\mathbf{A} = \mathrm{I}\,\mathbf{A}$, and we obtain again the fact mentioned above that $\mathrm{R}\,\mathrm{P}_{\mathrm{fin}}\,\mathbf{A} = \mathrm{P}_{\mathrm{fin}}\,\mathbf{A}$.

5.3 Boolean clones

Throughout this section, let $2 := \{0, 1\}$. When the base set A is 2, the relational clones are called Boolean relational clones. A complete list of clones of functions on 2 is known by the work of E. Post [Pos41]. In view of Theorem 1.6, this gives also a list of relational clones on 2. For each Boolean relational clone R' (up to duality, see below), a set R of relations generating R', i.e., $\mathrm{Cln}\,R = R'$, is given in the Table 5.1 below. The list of generating relations is based on [Lau02].

In this section, we choose for all Boolean relational clones R' a generating set R and determine the pseudovariety generated by the structure $(2, R)$. In view of Theorem 2.27, this is equivalent to give a set Σ of relational equations with the following property.

$$\begin{array}{l}(2, R) \models \Sigma \text{ and any finite structure } \mathbf{B} \text{ with } \mathbf{B} \models \Sigma \\ \text{satisfies all relational equations in } \mathrm{Th}_{\mathrm{re}}(2, R).\end{array} \tag{5.33}$$

In other words, $\mathrm{Mod}_{\mathrm{fin}}\,\Sigma = \mathrm{Mod}_{\mathrm{fin}}\,\mathrm{Th}_{\mathrm{re}}(2, R)$. In some cases, we even prove Property (5.33) for arbitrary (not only finite) structures $\mathbf{B}$. We choose R such that Σ can comfortably described. Often, this means a redundant R. If another generating set $\tilde{R}$ for the clone R' is given, then, by Lemma 3.7, Σ can be transformed into a set of relational equations with Property (5.33) for $\tilde{R}$. Especially, this applies to the case when $\tilde{R}$ is an irredundant subset of R.

We use the following relation symbols: $\mathsf{r}_0, \mathsf{r}_1$ (unary), r_c (binary), $\mathsf{r}_i, \mathsf{r}_s$ (ternary), r_u (4-ary), $\mathsf{r}_d^{(m)}$ (of arity m for $m = 2, 3, \ldots$), $\mathsf{r}_e^{(m)}, \mathsf{r}_o^{(m)}$ (of arity m for $m = 1, 2, \ldots$), and a binary symbol $\leq$ in infix notation.

We define an interpretation of the symbols above over the set 2. First, $\leq^{\mathbf{2}}$ is the canonical order on 2, determined by $0 \leq^{\mathbf{2}} 1$. Let sup and inf be the supremum and infimum function with respect to $\leq^{\mathbf{2}}$.

$$\begin{aligned}
\mathsf{r}_0^{\mathbf{2}} &:= \{\langle 0 \rangle\} \\
\mathsf{r}_1^{\mathbf{2}} &:= \{\langle 1 \rangle\} \\
\mathsf{r}_c^{\mathbf{2}} &:= \{\langle a, b \rangle \mid a \neq b\} \\
\mathsf{r}_i^{\mathbf{2}} &:= \{\langle a, b, c \rangle \mid a = \inf(b, c)\} \\
\mathsf{r}_s^{\mathbf{2}} &:= \{\langle a, b, c \rangle \mid a = \sup(b, c)\} \\
\mathsf{r}_u^{\mathbf{2}} &:= \{\langle a, b, c \rangle \mid a = b \text{ or } a = c\} \\
\mathsf{r}_d^{(m)\mathbf{2}} &:= A^m \setminus \{1\}^m && m = 2, 3, \ldots \\
\mathsf{r}_e^{(m)\mathbf{2}} &:= \{\langle a_0, \ldots, a_{m-1} \rangle \mid a_0 + \cdots + a_{m-1} = 0 \pmod 2\} && m = 1, 2, \ldots \\
\mathsf{r}_o^{(m)\mathbf{2}} &:= \{\langle a_0, \ldots, a_{m-1} \rangle \mid a_0 + \cdots + a_{m-1} = 1 \pmod 2\} && m = 1, 2, \ldots
\end{aligned}$$

We defined some relations twice, e.g. $\mathsf{r}_e^{(1)\mathbf{2}} = \mathsf{r}_0^{\mathbf{2}}$.

Giving a generating set R, we implicitly fix a type $\mathcal{R}$ for the discussion of the corresponding Σ, namely the set of all symbols r from the list above whose interpretations $\mathsf{r}^{\mathbf{2}}$ are in R.

We call two Boolean relational clones dual if they are obtained from each other by interchanging 0 and 1. Since for dual clones R_1' and R_2' we can choose generating sets R_1 and R_2 such that $(2, R_1)$ and $(2, R_2)$ are isomorphic, and isomorphic structures obey the same relational equations, we consider just one clone out of each pair of dual clones. In Table 5.1, we list the relational clones on $\{0, 1\}$ up to duality. For each clone R', the first column shows a name (according to naming scheme used in [PK79]), the second column shows a set R of relations generating R', the third column shows a irredundant subset of R (if possible), and the last column refers to the proposition where we determine Σ with Property (5.33).

Clone	generating set R	Basis	Σ given in
C_1	$\{\}$		
C_3	$\{\mathsf{r}_0\}$	$\{\mathsf{r}_0\}$	5.27
C_4	$\{\mathsf{r}_0, \mathsf{r}_1\}$	$\{\mathsf{r}_0, \mathsf{r}_1\}$	5.27
M_1	$\{\leq\}$	$\{\leq\}$	
M_3	$\{\leq, \mathsf{r}_0\}$	$\{\leq, \mathsf{r}_0\}$	5.28
M_4	$\{\leq, \mathsf{r}_0, \mathsf{r}_1\}$	$\{\leq, \mathsf{r}_0, \mathsf{r}_1\}$	5.28
$F_8^{(l)}$	$\{\mathsf{r}_0\} \cup \{\mathsf{r}_d^{(m)} \mid m = 2, \ldots, l\}$	$\{\mathsf{r}_d^{(l)}\}$	5.27
$F_8^{(\infty)}$	$\{\mathsf{r}_0\} \cup \{\mathsf{r}_d^{(m)} \mid m = 2, 3, \ldots\}$		5.27
$F_5^{(l)}$	$\{\mathsf{r}_0, \mathsf{r}_1\} \cup \{\mathsf{r}_d^{(m)} \mid m = 2, \ldots, l\}$	$\{\mathsf{r}_1, \mathsf{r}_d^{(l)}\}$	5.27
$F_5^{(\infty)}$	$\{\mathsf{r}_0, \mathsf{r}_1\} \cup \{\mathsf{r}_d^{(m)} \mid m = 2, 3, \ldots\}$		5.27
$F_7^{(l)}$	$\{\mathsf{r}_0, \leq\} \cup \{\mathsf{r}_d^{(m)} \mid m = 2, \ldots, l\}$	$\{\leq, \mathsf{r}_d^{(l)}\}$	5.28
$F_7^{(\infty)}$	$\{\mathsf{r}_0, \leq\} \cup \{\mathsf{r}_d^{(m)} \mid m = 2, 3, \ldots\}$		5.28
$F_6^{(l)}$	$\{\mathsf{r}_0, \mathsf{r}_1, \leq\} \cup \{\mathsf{r}_d^{(m)} \mid m = 2, \ldots, l\}$	$\{\mathsf{r}_1, \leq, \mathsf{r}_d^{(l)}\}$	5.28
$F_6^{(\infty)}$	$\{\mathsf{r}_0, \mathsf{r}_1, \leq\} \cup \{\mathsf{r}_d^{(m)} \mid m = 2, 3, \ldots\}$		5.28
D_3	$\{\mathsf{r}_c\}$	$\{\mathsf{r}_c\}$	5.30
D_1	$\{\mathsf{r}_c, \mathsf{r}_0, \mathsf{r}_1\}$	$\{\mathsf{r}_c, \mathsf{r}_0\}$	5.30
D_2	$\{\mathsf{r}_c, \leq, \mathsf{r}_0, \mathsf{r}_1\}$	$\{\mathsf{r}_c, \leq\}$	5.31
L_1	$\{\mathsf{r}_e^{(m)} \mid m = 2, 4, \ldots\}$	$\{\mathsf{r}_e^{(4)}\}$	5.33
L_3	$\{\mathsf{r}_e^{(m)} \mid m = 1, 2, \ldots\}$	$\{\mathsf{r}_e^{(3)}\}$	5.33
L_4	$\{\mathsf{r}_e^{(m)}, \mathsf{r}_o^{(m)} \mid m = 1, 2, \ldots\}$	$\{\mathsf{r}_e^{(1)}, \mathsf{r}_o^{(3)}\}$	5.33
L_5	$\{\mathsf{r}_e^{(m)}, \mathsf{r}_o^{(m)} \mid m = 2, 4, \ldots\}$	$\{\mathsf{r}_o^{(4)}\}$	5.33
P_6	$\{\mathsf{r}_i, \leq\}$	$\{\mathsf{r}_i\}$	5.29
P_3	$\{\mathsf{r}_i, \leq, \mathsf{r}_0\}$	$\{\mathsf{r}_i, \mathsf{r}_0\}$	5.29
P_5	$\{\mathsf{r}_i, \leq, \mathsf{r}_1\}$	$\{\mathsf{r}_i, \mathsf{r}_1\}$	5.29
P_1	$\{\mathsf{r}_i, \leq, \mathsf{r}_0, \mathsf{r}_1\}$	$\{\mathsf{r}_i, \mathsf{r}_0, \mathsf{r}_1\}$	5.29
O_9	$\{\mathsf{r}_u\}$	$\{\mathsf{r}_u\}$	5.34
O_8	$\{\mathsf{r}_i, \mathsf{r}_s, \leq\}$	$\{\mathsf{r}_i, \mathsf{r}_s\}$	5.32
O_6	$\{\mathsf{r}_i, \mathsf{r}_s, \leq, \mathsf{r}_0\}$	$\{\mathsf{r}_i, \mathsf{r}_s, \mathsf{r}_0\}$	5.32
O_4	$\{\mathsf{r}_u, \mathsf{r}_c\}$	$\{\mathsf{r}_u, \mathsf{r}_c\}$	5.34
O_1	$\{\mathsf{r}_i, \mathsf{r}_s, \leq, \mathsf{r}_0, \mathsf{r}_1\}$	$\{\mathsf{r}_i, \mathsf{r}_s, \mathsf{r}_0, \mathsf{r}_1\}$	5.32

Table 5.1: The Boolean clones

For sake of readability, we omit superscripts if the structure is clear from the context. Typical cases are

- Table 5.1, where we omit all superscripts $^{\mathbf{2}}$,
- definitions like $\mathbf{B} = (B, \mathsf{r})$ (instead of $\mathbf{B} = (B, \mathsf{r}^{\mathbf{B}})$),
- atomic formulas like $\mathsf{r}(b_1, b_2)$ (instead of $\mathsf{r}^{\mathbf{B}}(b_1, b_2)$) if it is clear that b_1, b_2 belong to $\mathbf{B}$.

Two gaps are left in the last column of Table 5.1. For C_1, it is easy to see that Σ can be chosen to be the empty set of relational equations. For M_1, a set Σ of relational equations was already given by Equations (5.1)–(5.4) and (5.7) which has the required properties by Proposition 5.6.

Given a map $\alpha\colon \mathbf{A} \to \mathbf{B}$, we say "$\alpha$ *preserves* r" if $\alpha(\mathsf{r}^{\mathbf{A}}) \subseteq \mathsf{r}^{\mathbf{B}}$. When presenting relational equations, we often denote by $\bar{x}$ a tuple of variables of appropriate length, distinct from all other occurring variables.

Before we examine the theories, it is appropriate to express a frequently encountered property of relations by relational equations.

Remark 5.26. Let r be a m-ary symbol, let $\mathbf{A}$ be any structure and $r = \mathsf{r}^{\mathbf{A}}$. There exists an $(m-1)$-ary function f such that

$$r = \{\langle f(\bar{a}), \bar{a}\rangle \mid \bar{a} \in A^{m-1}\}$$

if and only if $\mathbf{A}$ satisfies the relational equations

$$(\exists y)\ \mathsf{r}(y, \bar{x}) \leftrightarrow \mathsf{t}(\bar{x}),$$
$$\mathsf{r}(x, \bar{x}) \wedge \mathsf{r}(x', \bar{x}) \to x' \approx x.$$

We refer to this situation by saying "r is a function graph" or "r is the graph of f".

Given functions, equations involving these functions are equivalent to relational equations involving their graphs. For example, let $\mathsf{r}_1^{\mathbf{A}}$ and $\mathsf{r}_2^{\mathbf{A}}$ be the graphs of binary functions $f_1^{\mathbf{A}}$ and $f_2^{\mathbf{A}}$. Then in $\mathbf{A}$ it holds

$$f_1(x_0, f_2(x_1, x_0)) \approx f_1(x_0, x_1)$$

if and only if $\mathbf{A}$ satisfies

$$(\exists y_0, y_1)\ \mathsf{r}_2(y_0, x_1, x_0) \wedge \mathsf{r}_1(y_1, x_0, y_0) \wedge \mathsf{r}_1(y_1, x_0, x_1) \leftrightarrow \mathsf{t}(x_0, x_1).$$

We use this fact to abbreviate relational equations involving graphs of functions.

Because we process the large amount of data included in the description of the lattice of Boolean clones, some relational equations are clumsy, e.g. Equations 5.85–5.87. It is recommended to read such equations after reading the explanations given in the proofs which follow them.

Proposition 5.27. *Let* $\mathbf{2}$ *be the structure* $(2;\ \mathsf{r}_1, \mathsf{r}_0, \mathsf{r}_d^{(2)}, \mathsf{r}_d^{(3)}, \dots)$*. A set* Σ *with Property* (5.33) *for* $\mathbf{2}$ *is given by the following set of Equations* (5.34)–(5.45).

$$\mathsf{r}_0(x) \wedge \mathsf{r}_0(x') \to x \approx x' \tag{5.34}$$

$$(\exists y)\ \mathsf{r}_0(y) \leftrightarrow \mathsf{t} \tag{5.35}$$

$$\mathsf{r}_1(x) \wedge \mathsf{r}_1(x') \to x \approx x' \tag{5.36}$$

$$(\exists y)\ \mathsf{r}_1(y) \leftrightarrow \mathsf{t} \tag{5.37}$$

$$\mathsf{r}_0(x) \wedge \mathsf{r}_1(x) \leftrightarrow \mathsf{f}(x) \tag{5.38}$$

$$\mathsf{r}_d^{(m)} \text{ is totally symmetric} \qquad m = 2, 3, \dots \tag{5.39}$$

$$\mathsf{r}_d^{(m+1)}(x, x, \bar{x}) \leftrightarrow \mathsf{r}_d^{(m)}(x, \bar{x}) \qquad m = 2, 3, \dots \tag{5.40}$$

$$\mathsf{r}_d^{(2)}(x, x) \leftrightarrow \mathsf{r}_0(x) \tag{5.41}$$

$$\mathsf{r}_d^{(m)}(\bar{x}) \to \mathsf{r}_d^{(m+1)}(x, \bar{x}) \qquad m = 2, 3, \dots \tag{5.42}$$

$$\mathsf{r}_0(x_0) \to \mathsf{r}_d^{(2)}(x, x_0) \tag{5.43}$$

$$\mathsf{r}_1(x) \wedge \mathsf{r}_d^{(m+1)}(x, \bar{x}) \to \mathsf{r}_d^{(m)}(\bar{x}) \qquad m = 2, 3, \dots \tag{5.44}$$

$$\mathsf{r}_1(x) \wedge \mathsf{r}_d^{(2)}(x, x_0) \to \mathsf{r}_0(x_0) \tag{5.45}$$

Let $\mathbf{2}'$ *be one of the structures*

- $(2;\ \mathsf{r}_1, \mathsf{r}_0, \mathsf{r}_d^{(2)}, \dots, \mathsf{r}_d^{(l)})$, $l \in \mathbb{N}_+$, *or*
- $(2;\ \mathsf{r}_0, \mathsf{r}_d^{(2)}, \mathsf{r}_d^{(3)}, \dots)$, *or*
- $(2;\ \mathsf{r}_0, \mathsf{r}_d^{(2)}, \dots, \mathsf{r}_d^{(l)})$, $l \in \mathbb{N}_+$.

Then the set of relational equations in Σ *that contain only symbols occurring in* $\mathbf{2}'$ *has the Property* (5.33) *for* $\mathbf{2}'$.

Proof. Let $\mathbf{B}$ be a finite structure such that $\mathbf{B} \models \Sigma$. We show that $\mathbf{B} \in \mathrm{R\,P}\,\mathbf{2}$.

Let 0_B and 1_B be the unique elements in B satisfying $\mathsf{r}_0(0_B)$ and $\mathsf{r}_1(1_B)$ (cf. (5.34)–(5.37)). By (5.38), $0_B \neq 1_B$. We call $B' \subseteq B$ *overlapping* if $1_B \in B'$ and for all $m = 2, 3, \dots$ and $b_0, \dots, b_{m-1} \in B$

$$\mathsf{r}_d^{(m)}(b_0, \dots, b_{m-1}) \quad \text{implies} \quad \{b_0, \dots, b_{m-1}\} \nsubseteq B'.$$

By (5.41), this implies $0_B \notin B'$. Let $\mathrm{O}(\mathbf{B})$ be the set of all overlapping subsets of B. We are going to construct a retraction $(\alpha, \alpha')\colon \mathbf{2}^{\mathrm{O}(\mathbf{B})} \to \mathbf{B}$.

We define $\alpha'\colon \mathbf{B} \to \mathbf{2}^{\mathrm{O}(\mathbf{B})}$ by

$$\alpha'(b)(B') := \begin{cases} 1 & \text{if } b \in B' \\ 0 & \text{otherwise} \end{cases}, \qquad b \in B, B' \in \mathrm{O}(\mathbf{B}).$$

By construction, α' preserves r_1, r_0 and $\mathsf{r}_d^{(m)}$. Thus, α' is a homomorphism.

For all $b \neq 0_B$ we have $\{b, 1_B\} \in \mathrm{O}(\mathbf{B})$ (by (5.39)–(5.41),(5.44),(5.45)). Hence $\mathrm{O}(\mathbf{B})$ is nonempty and α' is injective. So we can define $\alpha\colon \mathbf{2}^{\mathrm{O}(\mathbf{B})} \to \mathbf{B}$ by

$$\alpha(a) := \begin{cases} b & \text{if } a = \alpha'(b) \\ 0_B & \text{otherwise} \end{cases}, \qquad a \in \mathbf{2}^{\mathrm{O}(\mathbf{B})}.$$

By construction, α preserves r_1 and r_0 and $\alpha\alpha' = \mathrm{id}_B$. It remains to show that α preserves the $\mathsf{r}_d^{(m)}$. Let $a_0, \ldots, a_{m-1} \in \mathbf{2}^{\mathrm{O}(\mathbf{B})}$ such that $\mathsf{r}_d^{(m)}(a_0, \ldots, a_{m-1})$, and $b_i := \alpha(a_i)$. If one of the b_i is 0_B then $\mathsf{r}_d^{(m)}(b_0, \ldots, b_{m-1})$ (by (5.42),(5.43)). So we can assume $a_i = \alpha'(b_i)$ and $b_i \neq 0_B$, $i \in \{0, \ldots, m-1\}$. We consider

$$B' := \{b_0, \ldots, b_{m-1}, 1_B\}.$$

We have $B' \notin \mathrm{O}(\mathbf{B})$. Indeed, $B' \in \mathrm{O}(\mathbf{B})$ would imply $a_i(B') = 1$ contradicting $\mathsf{r}_d^{(m)}(a_0, \ldots, a_{m-1})$. Thus, for some m' and $b_{i_1}, \ldots, b_{i_{m'}} \in B'$ it holds

$$\mathsf{r}_d^{(m')}(b_{i_1}, \ldots, b_{i_{m'}}).$$

Now, from (5.39)–(5.45), it follows

$$\mathsf{r}_d^{(m)}(b_0, \ldots, b_{m-1}).$$

Hence, $(\alpha, \alpha')\colon \mathbf{2}^{\mathrm{O}(\mathbf{B})} \to \mathbf{B}$ is a retraction. This completes the proof of that Σ has the Property (5.33) for $\mathbf{2}$.

To prove the assertion about $\mathbf{2}'$, we just have to restrict all arguments to the relations occurring in $\mathbf{2}'$. When $\mathbf{2}'$ does not contain the base relation r_1, $\mathrm{O}(\mathbf{B})$ can be empty; but this is the trivial case $|B| = 1$. $\square$

Proposition 5.28. *Let* $\mathbf{2}$ *be the structure* $(2;\ \leq, \mathsf{r}_1, \mathsf{r}_0, \mathsf{r}_d^{(2)}, \mathsf{r}_d^{(3)}, \ldots)$. *A set* Σ *with Property* (5.33) *for* $\mathbf{2}$ *is given by Equations* (5.34)–(5.45) *and the following set of Equations* (5.46)–(5.51).

$$\leq \textit{ is an sdor} \tag{5.46}$$
$$\mathsf{r}_0(x_0) \to x_0 \leq x_1 \tag{5.47}$$
$$\mathsf{r}_1(x_0) \to x_1 \leq x_0 \tag{5.48}$$
$$x \leq \{\bar{x}\} \wedge \mathsf{r}_d^{(m)}(\bar{x}) \to \mathsf{r}_0(x) \qquad m = 2, 3, \ldots \tag{5.49}$$
$$\mathsf{r}_d^{(m)}(x, \bar{x}) \wedge x' \leq x \to \mathsf{r}_d^{(m)}(x', \bar{x}) \qquad m = 2, 3, \ldots \tag{5.50}$$
$$\mathsf{r}_d^{(m)}(x, \bar{x}) \wedge \mathsf{r}_d^{(m)}(x', \bar{x}) \to (\exists y)\ y \geq \{x, x'\} \wedge \mathsf{r}_d^{(m)}(y, \bar{x}) \qquad m = 2, 3, \ldots \tag{5.51}$$

Let $\mathbf{2}'$ *be one of the structures*

- $(2;\ \leq, \mathsf{r}_1, \mathsf{r}_0, \mathsf{r}_d^{(2)}, \ldots, \mathsf{r}_d^{(l)})$, $l \in \mathbb{N}_+$, *or*
- $(2;\ \leq, \mathsf{r}_0, \mathsf{r}_d^{(2)}, \mathsf{r}_d^{(3)}, \ldots)$, *or*
- $(2;\ \leq, \mathsf{r}_0, \mathsf{r}_d^{(2)}, \ldots, \mathsf{r}_d^{(l)})$, $l \in \mathbb{N}_+$.

Then the set of relational equations in Σ *that contain only symbols occurring in* $\mathbf{2}'$ *has the Property* (5.33) *for* $\mathbf{2}'$.

Proof. Let $\mathbf{B}$ be a finite structure such that $\mathbf{B} \models \Sigma$. We show that $\mathbf{B} \in \mathrm{R}\,\mathrm{P}\,\mathbf{2}$.

Let 0_B and 1_B be the unique elements in B satisfying $\mathsf{r}_0(0_B)$ and $\mathsf{r}_1(1_B)$ (cf. (5.34)–(5.37)). By (5.38), $0_B \neq 1_B$. We call $B' \subseteq B$ an *upper segment* if $b \in B'$

and $b \leq b'$ implies $b' \in B'$. Given an arbitrary $B'' \subseteq B$, the upper segment generated by B'' is the set

$$\{b' \in B \mid \exists b \in B''\ b \leq b'\}$$

Let $\mathrm{UO}(\mathbf{B})$ be the set of all overlapping (cf. proof of Proposition 5.27) upper segments of B. We are going to construct a retraction $(\alpha, \alpha')\colon \mathbf{2}^{\mathrm{UO}(\mathbf{B})} \to \mathbf{B}$.

We define $\alpha'\colon \mathbf{B} \to \mathbf{2}^{\mathrm{UO}(\mathbf{B})}$ by

$$\alpha'(b)(B') := \begin{cases} 1 & \text{if } b \in B' \\ 0 & \text{otherwise,} \end{cases} \qquad b \in B, B' \in \mathrm{UO}(\mathbf{B}).$$

By construction, α' preserves $\leq$, r_1, r_0 and $\mathsf{r}_d^{(m)}$. Thus, α' is a homomorphism.

Since $(B, \leq)$ is a finite sdor, it is a lattice. Let sup be the supremum function with respect to this lattice. We define $\alpha\colon \mathbf{2}^{\mathrm{UO}(\mathbf{B})} \to \mathbf{B}$ by

$$\alpha(a) := \sup\{b \in B \mid \alpha'(b) \leq a\}, \qquad a \in \mathbf{2}^{\mathrm{UO}(\mathbf{B})}.$$

Clearly, α preserves r_1 and $\leq$. For all $b \neq 0_B$ we have that the upper segment generated by b is in $\mathrm{UO}(\mathbf{B})$ (by (5.48), (5.49)). Hence, $\mathrm{UO}(\mathbf{B})$ is nonempty and α preserves r_0. To prove that α is a homomorphism, it remains to show the following claim.

Claim (*). Let $a_0, \dots, a_{m-1} \in \mathbf{2}^{\mathrm{UO}(\mathbf{B})}$ such that $\mathsf{r}_d^{(m)}(a_0, \dots, a_{m-1})$. Then $\mathsf{r}_d^{(m)}(\alpha(a_0), \dots, \alpha(a_{m-1}))$.

Let $b_0, \dots, b_{m-1} \in B$ such that $\alpha'(b_i) \leq a_i$, $i \in \{0, \dots, m-1\}$. We consider the upper segment B' generated by $\{b_0, \dots, b_{m-1}\}$. We have $B' \notin \mathrm{UO}(\mathbf{B})$. Indeed, $B' \in \mathrm{UO}(\mathbf{B})$ would imply $a_i(B') \geq \alpha'(b_i)(B') = 1$ contradicting $\mathsf{r}_d^{(m)}(a_0, \dots, a_{m-1})$. Thus, for some m' and $b'_0, \dots, b'_{m'-1} \in B'$ it holds

$$\mathsf{r}_d^{(m')}(b'_0, \dots, b'_{m'-1}).$$

Now, from (5.39)–(5.45) and (5.50), it follows $\mathsf{r}_d^{(m)}(b_0, \dots, b_{m-1})$. By (5.50) and (5.51),

$$\mathsf{r}_d^{(m)}(\alpha(a_0), \dots, \alpha(a_{m-1})).$$

This completes the proof of Claim (*).

Claim (**). For all $b \in B$, it holds $\alpha(\alpha'(b)) = b$.

Let $b, b' \in B$. We have already observed that $b' \leq b$ implies $\alpha'(b') \leq \alpha'(b)$. To show the converse, assume $\alpha'(b') \leq \alpha'(b)$. If $b' = 0_B$ then $b' \leq b$. If $b' \neq 0_B$ then the upper segment B' generated by $\{b'\}$ is in $\mathrm{UO}(\mathbf{B})$, hence, $b' \leq b$. Thus we have

$$\alpha(\alpha'(b)) = \sup\{b' \in B \mid \alpha'(b') \leq \alpha'(b)\} = \sup\{b' \in B \mid b' \leq b\} = b.$$

This completes the proof of Claim (**).

Hence, $(\alpha, \alpha')\colon \mathbf{2}^{\mathrm{UO}(\mathbf{B})} \to \mathbf{B}$ is a retraction. This completes the proof of that Σ has the Property (5.33) for $\mathbf{2}$.

To prove the assertion about $\mathbf{2}'$, we just have to restrict all arguments to the relations occurring in $\mathbf{2}'$. When $\mathbf{2}'$ does not contain the base relation r_1, $\mathrm{UO}(\mathbf{B})$ can be empty; but this is the trivial case $|B| = 1$. $\square$

Proposition 5.29. *Let* $\mathbf{2}$ *be the structure* $(2;\ \leq, \mathsf{r}_i, \mathsf{r}_0, \mathsf{r}_1)$. *A set* Σ *with Property* (5.33) *for* $\mathbf{2}$ *is given by Equations* (5.34)–(5.38), (5.46)–(5.48) *and the following set of Equations* (5.52)–(5.55).

$$\mathsf{r}_i \textit{ is a function graph} \tag{5.52}$$

$$\mathsf{r}_i(x, x_0, x_1) \to x \leq \{x_0, x_1\} \tag{5.53}$$

$$x' \leq \{x_0, x_1\} \wedge \mathsf{r}_i(x, x_0, x_1) \to x' \leq x \tag{5.54}$$

This implies that for any structure $\mathbf{B}$ *satisfying* Σ *we have that* $(B, \leq)$ *is a* inf*-semilattice order, and* $\mathsf{r}_i^{\mathbf{B}}$ *is the graph of the* inf*-function. For convenience, we formulate the remaining relational equation in terms of the function* inf *(cf. Remark 5.26).*

$$\inf(x_1, x_2) \leq x_0 \wedge \inf(x_1, x_3) \leq x_0 \to (\exists y)\ \{x_2, x_3\} \leq y \wedge \inf(y, x_1) \leq x_0 \tag{5.55}$$

Let $\mathbf{2}'$ *be one of the structures* $(2;\ \leq, \mathsf{r}_i, \mathsf{r}_0)$, *or* $(2;\ \leq, \mathsf{r}_i, \mathsf{r}_1)$, *or* $(2;\ \leq, \mathsf{r}_i)$. *Then the set of relational equations in* Σ *that contain only symbols occurring in* $\mathbf{2}'$ *has the Property* (5.33) *for* $\mathbf{2}'$.

Proof. Let $\mathbf{B}$ be a finite structure such that $\mathbf{B} \models \Sigma$. We show that $\mathbf{B} \in \mathrm{R\,P}\,\mathbf{2}$.

Let 0_B and 1_B be the unique elements in B satisfying $\mathsf{r}_0(0_B)$ and $\mathsf{r}_1(1_B)$ (cf. (5.34)–(5.37)). By (5.38), $0_B \neq 1_B$. Since $\mathbf{B}$ is finite, (5.46) implies that $(B, \leq)$ is a lattice order. Equation (5.55) is that this lattice is distributive. By (5.47), (5.48), 0_B is the smallest element and 1_B is the largest element.

We call an element $b \in B$ inf-irreducible if for all $b_0, b_1 \in B$ we have $b = \inf(b_0, b_1)$ implies $b = b_0$ or $b = b_1$. Let $\mathrm{I}(\mathbf{B})$ be the set of inf-irreducible elements of $(B, \leq)$ except 1_B. We are going to construct a retraction $(\alpha, \alpha')\colon \mathbf{2}^{\mathrm{I}(\mathbf{B})} \to \mathbf{B}$.

We define $\alpha'\colon \mathbf{B} \to \mathbf{2}^{\mathrm{I}(\mathbf{B})}$ by

$$\alpha'(b)(b') := \begin{cases} 0 & \text{if } b \leq b' \\ 1 & \text{otherwise} \end{cases}, \qquad b \in B, b' \in \mathrm{I}(\mathbf{B}).$$

By construction, α' preserves r_1, r_0 and $\leq$. It is well known, that in a distributive lattice we have $\inf(b_0, b_1) \leq b'$ if and only if $b_0 \leq b'$ or $b_1 \leq b'$, where b' is a inf-irreducible element and b_0 and b_1 are arbitrary elements (see, e.g. [MMT87]). Thus, α' preserves r_i and is a homomorphism.

We define $\alpha\colon \mathbf{2}^{\mathrm{I}(\mathbf{B})} \to \mathbf{B}$ by

$$\alpha(a) := \inf\{b' \in \mathrm{I}(\mathbf{B}) \mid a(b') = 0\}, \qquad a \in 2^{\mathrm{I}(\mathbf{B})}.$$

It is straightforward to check that α is a homomorphism. For all $b \in B$ it holds

$$\alpha(\alpha'(b)) = \inf\{b' \in \mathrm{I}(\mathbf{B}) \mid \alpha'(b)(b') = 0\} = \inf\{b' \in \mathrm{I}(\mathbf{B}) \mid b \leq b'\} = b.$$

Hence, $(\alpha, \alpha')\colon \mathbf{2}^{\mathrm{I}(\mathbf{B})} \to \mathbf{B}$ is a retraction. This completes the proof of that Σ has the Property (5.33) for $\mathbf{2}$.

To prove the assertion about $\mathbf{2}'$, we just have to restrict all arguments to the relations occurring in $\mathbf{2}'$. When $\mathbf{2}'$ does not contain the base relation r_0, $\mathrm{I}(\mathbf{B})$ can be empty; but this is the trivial case $|B| = 1$. $\square$

Proposition 5.30. *Let* $\mathbf{2}$ *be the structure* $(2;\ \mathsf{r}_c, \mathsf{r}_0, \mathsf{r}_1)$*. A set* Σ *with Property* (5.33) *for* $\mathbf{2}$ *is given by Equations* (5.34)–(5.38) *and the following set of Equations* (5.56)–(5.60)*.*

$$\mathsf{r}_c \text{ is a function graph} \tag{5.56}$$
$$\mathsf{r}_c(x, x) \to \mathsf{f}(x) \tag{5.57}$$

Let c *denote the unary function with graph* r_c*. For convenience, we formulate the remaining relational equations in terms of the function* c *(cf. Remark 5.26).*

$$c(c(x)) \approx x \tag{5.58}$$

and

$$\mathsf{r}_0(x) \to \mathsf{r}_1(c(x)) \tag{5.59}$$
$$\mathsf{r}_1(x) \to \mathsf{r}_0(c(x)) \tag{5.60}$$

Let $\mathbf{2}'$ *be the structure* $(2;\ \mathsf{r}_c)$*. Then the set of relational equations in* Σ *that contain only symbols occurring in* $\mathbf{2}'$ *has the Property* (5.33) *for* $\mathbf{2}'$*.*

Proof. Let $\mathbf{B}$ be a finite structure such that $\mathbf{B} \models \Sigma$. Choose $n \in \mathbb{N}_+$ such that $2^n \geq |B|$. Now it is easy to find a retraction from $\mathbf{2}^n$ ($\mathbf{2}'^n$ resp.) to $\mathbf{B}$. $\square$

Proposition 5.31. *Let* $\mathbf{2}$ *be the structure* $(2;\ \leq, \mathsf{r}_c, \mathsf{r}_0, \mathsf{r}_1)$*. A set* Σ *with Property* (5.33) *for* $\mathbf{2}$ *is given by Equations* (5.34)–(5.38)*,* (5.46)–(5.48)*,* (5.56)–(5.60)*, and the following set of Equations* (5.61)–(5.63)*. Let* c *denote the unary function with graph* r_c*. For convenience, we formulate the remaining relational equations in terms of the function* c *(cf. Remark 5.26).*

$$x \leq x_0 \wedge x \leq c(x_0) \to \mathsf{r}_0(x) \tag{5.61}$$
$$x_0 \leq x \wedge c(x_0) \leq x \to \mathsf{r}_1(x) \tag{5.62}$$
$$x_0 \leq x_1 \to c(x_1) \leq c(x_0) \tag{5.63}$$

Proof. Let $\mathbf{B}$ be a finite structure such that $\mathbf{B} \models \Sigma$. We show that $\mathbf{B} \in \mathrm{R\,P}\,\mathbf{2}$.

Let 0_B and 1_B be the unique elements in B satisfying $\mathsf{r}_0(0_B)$ and $\mathsf{r}_1(1_B)$ (cf. (5.34)–(5.37)). By (5.38), $0_B \neq 1_B$. We call $B' \subseteq B$ an upper segment if $b' \in B'$ and $b' \leq b$ implies $b \in B'$. We call $B' \subseteq B$ complementary if for all $b \in B$ exactly one of the elements b and $c(b)$ belong to B' (cf. (5.56)–(5.58)). Let $\mathrm{UC}(\mathbf{B})$ be the set of all complementary upper segments. By (5.47),(5.48),(5.59),(5.60), for all $B' \in \mathrm{UC}(\mathbf{B})$ it holds $0_B \notin B'$ and $1_B \in B'$. We are going to construct a retraction $(\alpha, \alpha')\colon \mathbf{2}^{\mathrm{UC}(\mathbf{B})} \to \mathbf{B}$.

We define $\alpha'\colon \mathbf{B} \to \mathbf{2}^{\mathrm{UC}(\mathbf{B})}$ by

$$\alpha'(b)(B') := \begin{cases} 1 & \text{if } b \in B' \\ 0 & \text{otherwise} \end{cases}, \qquad b \in B, B' \in \mathrm{UC}(\mathbf{B}).$$

By construction, α' preserves $\leq$, r_c, r_0 and r_1, thus α' is a homomorphism.

For the construction of α we need the following two claims.

Claim (*). Let $B' \subseteq B$ be a nonempty upper segment with the property

$$\text{for all } b' \in B \text{ at most one of the elements } b' \text{ and } c(b') \text{ belong to } B'. \qquad (*)$$

If $b \in B$ is such that $b, c(b) \notin B'$ then $B'' := B' \cup \{b'' \in B \mid b \leq b''\}$ satisfies Property (*). Hence, any upper segment B' with Property (*) can be extended to an element of $\mathrm{UC}(\mathbf{B})$.

Assume, to the contrary, that there is a $b' \in B$ with $b', c(b') \in B''$. By the assumption of the claim it can not hold $b', c(b') \in B'$. If it where $b \leq b', c(b')$, we would have $b = 0_B$ (by (5.61)), $c(b) = 1_B$ (by (5.59)) and, by (5.48), $c(b) \in B'$, a contradiction. Without loss of generality, it remains the case $b' \in B'$ and $b \leq c(b')$. This implies $b' \leq c(b)$ (by (5.63)) and $c(b) \in B'$, a contradiction. This completes the proof of Claim (*).

Claim (**). For all $b_0, b_1 \in B$ we have $\alpha'(b_0) \leq \alpha'(b_1)$ if and only if $b_0 \leq b_1$.

We have already observed that α' preserves $\leq$. Let $\alpha'(b_0) \leq \alpha'(b_1)$ and assume, to the contrary, $b_0 \not\leq b_1$. By (5.61) and (5.47), the upper segment B' generated by b_0 has the Property (*). If $c(b_1) \notin B'$, we can extend B' to an upper segment B'' with Property (*) such that $c(b_1) \in B''$ (by Claim (*)). Using once more Claim(*) we obtain a $B''' \in \mathrm{UC}(\mathbf{B})$ satisfying $b_0, c(b_1) \in B'''$, thus $b_1 \notin B'''$. It follows

$$\alpha'(b_0)(B''') \not\leq \alpha'(b_1)(B'''),$$

a contradiction. This completes the proof of Claim (**).

By Claim (**), α' is injective. We define a partial map $\alpha_0 \colon \mathbf{2}^{\mathrm{UC}(\mathbf{B})} \to \mathbf{B}$ by $\alpha_0(\alpha'(b)) = b$ for all $b \in B$. Obviously, α_0 preserves r_c, r_0 and r_1. By Claim (**), α_0 preserves $\leq$. Note that if $\alpha_0(a)$ is defined then so is $\alpha_0(c(a))$. Now we extend α_0 inductively as follows: Let a be any element of $2^{\mathrm{UC}(\mathbf{B})}$ such that $\alpha_{i-1}(a)$ and $\alpha_{i-1}(c(a))$ are not already defined. Since $(B, \leq)$ is a finite sdor, it is a lattice. Let sup be the supremum function with respect to this lattice. We extend α_{i-1} to α_i by setting

$$\alpha_i(a) := \sup\{\alpha_{i-1}(a') \mid a' \leq a, \alpha_{i-1}(a') \text{ is defined}\} \quad \text{and} \quad \alpha_i(c(a)) := c(\alpha_i(a)).$$

By construction, α_i preserves r_c, r_0 and r_1. To show that α_i preserves $\leq$ we consider three cases. Let a_1 be such that $\alpha_{i-1}(a_1)$ is defined.

- Let $a \leq a_1$. Then $\alpha_{i-1}(a') \leq \alpha_{i-1}(a_1)$ for all a' occurring in the Definition of $\alpha_i(a)$, by the induction hypothesis. Hence $\alpha_i(a) \leq \alpha_i(a_1)$.
- Let $a \geq a_1$. Then a_1 is one of the a' occurring in the Definition of $\alpha_i(a)$, thus, $\alpha_i(a) \geq \alpha_i(a_1)$.
- Let $c(a) \leq a_1$. Then $a \geq c(a_1)$, hence $\alpha_i(a) \geq \alpha_i(c(a_1))$ and, by (5.63), $c(\alpha_i(a)) \leq c(\alpha_i(c(a_1)))$. Since α_i preserves r_c we have $\alpha_i(c(a)) \leq \alpha_i(a_1)$. The same argument works with $\leq$ and $\geq$ interchanged.

The cases $a \leq c(a)$ and $c(a) \leq a$ can not occur. Indeed, such an inequality would imply that $\alpha_0(a)$ and $\alpha_0(c(a))$ are already defined. Thus, α_i preserves $\leq$.

Since $\mathbf{2}^{\mathrm{UC}(\mathbf{B})}$ is finite, we obtain a homomorphism $\alpha\colon \mathbf{2}^{\mathrm{UC}(\mathbf{B})} \to \mathbf{B}$. By construction, for all $b \in B$ it holds $\alpha(\alpha'(b)) = b$. Hence, $(\alpha, \alpha')\colon \mathbf{2}^{\mathrm{UC}(\mathbf{B})} \to \mathbf{B}$ is a retraction. This completes the proof. $\square$

Proposition 5.32. *Let* $\mathbf{2}$ *be the structure* $(2;\ \leq, \mathsf{r}_i, \mathsf{r}_s, \mathsf{r}_0, \mathsf{r}_1)$*. A set* Σ *with Property* (5.33) *for* $\mathbf{2}$ *is given by Equations* (5.34)–(5.38)*,* (5.46)–(5.48)*,* (5.52)–(5.54) *and the following set of Equations* (5.64)–(5.69)*.*

$$\mathsf{r}_s \text{ is a function graph} \tag{5.64}$$
$$\mathsf{r}_s(x, x_0, x_1) \to \{x_0, x_1\} \leq x \tag{5.65}$$
$$\{x_0, x_1\} \leq x \wedge \mathsf{r}_s(x', x_0, x_1) \to x' \leq x \tag{5.66}$$

This implies that for any structure $\mathbf{B}$ *satisfying* Σ *we have that* $(B, \leq)$ *is a lattice order, and* r_s *(*r_i *resp.) is the graph of the* sup*-function (*inf*-function resp.). For convenience, we formulate the remaining relational equations in terms of the functions* sup *and* inf *(cf. Remark 5.26).*

$$\inf(x_0, \sup(x_1, x_2)) \approx \sup(\inf(x_0, x_1), \inf(x_0, x_2)) \tag{5.67}$$
$$\sup(x_0, \inf(x_1, x_2)) \approx \inf(\sup(x_0, x_1), \sup(x_0, x_2)) \tag{5.68}$$

and

$$x_0 \leq x_1 \leq x_2 \to (\exists y)\ \inf(x_1, y) \approx x_0 \wedge \sup(x_1, y) \approx x_2. \tag{5.69}$$

Let $\mathbf{2}'$ *be one of the structures* $(2;\ \leq, \mathsf{r}_i, \mathsf{r}_s, \mathsf{r}_0)$*, or* $(2;\ \leq, \mathsf{r}_i, \mathsf{r}_s, \mathsf{r}_1)$*, or* $(2;\ \leq, \mathsf{r}_i, \mathsf{r}_s)$*. Then the set of relational equations in* Σ *that contain only symbols occurring in* $\mathbf{2}'$ *has the Property* (5.33) *for* $\mathbf{2}'$*.*

Proof. Let $\mathbf{B}$ be a finite structure such that $\mathbf{B} \models \Sigma$. We need to show that $\mathbf{B} \in \mathrm{R\,P}\,\mathbf{2}$. We even show that $\mathbf{B} \in \mathrm{I\,P}\,\mathbf{2}$.

Let 0_B and 1_B be the unique elements in B satisfying $\mathsf{r}_0(0_B)$ and $\mathsf{r}_1(1_B)$ (cf. (5.34)–(5.37)). By (5.38), $0_B \neq 1_B$. We summarize (5.46)–(5.48), (5.52)–(5.54) and (5.64)–(5.66) in saying that $(B, \leq)$ is a lattice order, r_s (r_i resp.) is the graph of the sup-function (inf-function resp.), and 0_B is the smallest element and 1_B is the largest element. Equations (5.67) and (5.68) say that this lattice is distributive. Equation (5.69) is that this lattice is complemented, hence, it is a Boolean lattice. It is well known that such a $\mathbf{B}$ is isomorphic to $\mathbf{2}^I$, for some finite set I.

To prove the assertion about $\mathrm{Th}_{\mathrm{re}}\,\mathbf{2}'$, we just have to restrict all arguments to the relations occurring in $\mathbf{2}'$. $\square$

Proposition 5.33. *Let* $\mathbf{2}$ *be the structure* $(2;\ \mathsf{r}_e^{(1)}, \mathsf{r}_e^{(2)}, \ldots, \mathsf{r}_o^{(1)}, \mathsf{r}_o^{(2)}, \ldots)$*. A set* Σ *with Property* (5.33) *for* $\mathbf{2}$ *is given by the following set of Equations* (5.70)–(5.78)*.*

$$\mathsf{r}_e^{(m)} \text{ is totally symmetric} \qquad m = 2, 3, \ldots \tag{5.70}$$
$$\mathsf{r}_o^{(m)} \text{ is totally symmetric} \qquad m = 2, 3, \ldots \tag{5.71}$$

$$\mathsf{r}_e^{(2)}(x,x) \leftrightarrow \mathsf{t}(x) \tag{5.72}$$

$$\mathsf{r}_o^{(2)}(x,x) \leftrightarrow \mathsf{f}(x) \tag{5.73}$$

$$(\exists y)\ \mathsf{r}_e^{(m)}(y,\bar{x}) \leftrightarrow \mathsf{t}(\bar{x}) \qquad m = 1,2,\dots \tag{5.74}$$

$$(\exists y)\ \mathsf{r}_o^{(m)}(y,\bar{x}) \leftrightarrow \mathsf{t}(\bar{x}) \qquad m = 1,2,\dots \tag{5.75}$$

$$\mathsf{r}_e^{(m)}(\bar{x}) \wedge \mathsf{r}_e^{(m')}(\bar{x}') \leftrightarrow \mathsf{r}_e^{(m)}(\bar{x}) \wedge \mathsf{r}_e^{(m+m')}(\bar{x},\bar{x}') \qquad m,m' = 1,2,\dots \tag{5.76}$$

$$\mathsf{r}_e^{(m)}(\bar{x}) \wedge \mathsf{r}_o^{(m')}(\bar{x}') \leftrightarrow \mathsf{r}_e^{(m)}(\bar{x}) \wedge \mathsf{r}_o^{(m+m')}(\bar{x},\bar{x}') \qquad m,m' = 1,2,\dots \tag{5.77}$$

$$\mathsf{r}_o^{(m)}(\bar{x}) \wedge \mathsf{r}_o^{(m')}(\bar{x}') \leftrightarrow \mathsf{r}_o^{(m)}(\bar{x}) \wedge \mathsf{r}_e^{(m+m')}(\bar{x},\bar{x}') \qquad m,m' = 1,2,\dots \tag{5.78}$$

Let $\mathbf{2}'$ *be one of the structures*

- $(2;\ \mathsf{r}_e^{(2)},\mathsf{r}_e^{(4)},\dots,\mathsf{r}_o^{(2)},\mathsf{r}_o^{(4)},\dots)$, *or*
- $(2;\ \mathsf{r}_e^{(1)},\mathsf{r}_e^{(2)},\dots)$, *or*
- $(2;\ \mathsf{r}_e^{(2)},\mathsf{r}_e^{(4)},\dots)$.

Then the set of relational equations in Σ *that contain only symbols occurring in* $\mathbf{2}'$ *has the Property* (5.33) *for* $\mathbf{2}'$.

Proof. Let $\mathbf{B}$ be a finite structure such that $\mathbf{B} \models \Sigma$. In contrast to the foregoing propositions, we use syntactical arguments. Let φ_1, φ_2 be primitive-positive formulas such that $\mathbf{2} \models \varphi_1 \leftrightarrow \varphi_2$. We have to show $\mathbf{B} \models \varphi_1 \leftrightarrow \varphi_2$.

First, we observe that from (5.70),(5.71) and (5.74)–(5.78) it follows that for any primitive-positive formula φ we can find a quantifier-free primitive-positive formula φ' such that it holds $\varphi^{\mathbf{B}} = \varphi'^{\mathbf{B}}$. Hence, we can assume φ_1 and φ_2 to be quantifier-free.

We consider $\varphi^{\mathbf{2}}$ for any quantifier-free primitive-positive formula φ. By definition, it holds $\bar{a} \in \mathsf{r}_e^{(m)\mathbf{2}}$ ($\bar{a} \in \mathsf{r}_o^{(m)\mathbf{2}}$ resp.) if and only if $\bar{a}$ is the solution of a homogeneous (nonhomogeneous resp.) linear equation over the two-element field. Thus, $\bar{a} \in \varphi^{\mathbf{2}}$ if and only if $\bar{a}$ is the solution of a system of linear equations over the two-element field. Hence, $\mathbf{2} \models \varphi_1 \leftrightarrow \varphi_2$ is equivalent to that φ_1 and φ_2, regarded as systems of linear equations, have the same solutions.

It is well known that two systems of linear equations over the two-element field have the same solutions if and only if one can be obtained from the other by the formation linear combinations, i.e., by replacing two linear equations of the form

$$\sum x_i = a \qquad \text{and} \qquad \sum x_i' = a'$$

by

$$\sum x_i = a \qquad \text{and} \qquad \sum x_i + \sum x_i' = a + a'.$$

This implies, by (5.70),(5.73) and (5.76)–(5.78), that we can find primitive-positive formulas φ_1' and φ_2' which are syntactically equal such that $\varphi_1^{\mathbf{B}} = {\varphi_1'}^{\mathbf{B}}$ and $\varphi_2^{\mathbf{B}} = {\varphi_2'}^{\mathbf{B}}$. Thus, $\varphi_1^{\mathbf{B}} = \varphi_2^{\mathbf{B}}$, i.e., $\mathbf{B} \models \varphi_1 \leftrightarrow \varphi_2$. $\square$

Proposition 5.34. *Let* $\mathbf{2}$ *be the structure* $(2;\ \mathsf{r}_u)$*. A set* Σ *with Property* (5.33) *for* $\mathbf{2}$ *is given by the following set of Equations* (5.79)–(5.87)*. We abbreviate two primitive-positive formulas as follows.*

$$\varphi_{\inf}(x_0, x_1, x_2, x_3) := \mathsf{r}_u(x_1, x_0, x_2) \wedge \mathsf{r}_u(x_1, x_0, x_3) \wedge \mathsf{r}_u(x_1, x_2, x_3)$$
$$\varphi_{\sup}(x_0, x_1, x_2, x_3) := \mathsf{r}_u(x_2, x_0, x_1) \wedge \mathsf{r}_u(x_3, x_0, x_1) \wedge \mathsf{r}_u(x_1, x_2, x_3)$$

For convenience, we formulate the remaining relational equations using these abbreviations.

$$\mathsf{r}_u(x_1, x_0, x_2) \wedge \mathsf{r}_u(x_2, x_0, x_1) \leftrightarrow x_1 \approx x_2 \tag{5.79}$$
$$\mathsf{r}_u(x_1, x_0, x_2) \wedge \mathsf{r}_u(x_2, x_0, x_3) \rightarrow \mathsf{r}_u(x_1, x_0, x_3) \tag{5.80}$$
$$(\exists y)\ \varphi_{\inf}(x_0, y, x_1, x_2) \leftrightarrow \mathsf{t}(x_0, x_1, x_2) \tag{5.81}$$
$$\varphi_{\inf}(x_0, x_1, x_2, x_3) \wedge \mathsf{r}_u(x_4, x_0, x_2) \wedge \mathsf{r}_u(x_4, x_0, x_3) \rightarrow \mathsf{r}_u(x_4, x_0, x_1) \tag{5.82}$$
$$(\exists y)\ \varphi_{\sup}(x_0, y, x_1, x_2) \leftrightarrow \mathsf{t}(x_0, x_1, x_2) \tag{5.83}$$
$$\varphi_{\sup}(x_0, x_1, x_2, x_3) \wedge \mathsf{r}_u(x_2, x_0, x_4) \wedge \mathsf{r}_u(x_3, x_0, x_4) \rightarrow \mathsf{r}_u(x_1, x_0, x_4) \tag{5.84}$$

$$\begin{aligned}(\exists y_1, y_2)\ \varphi_{\inf}(x_0, y_1, x_1, x_2) \wedge \varphi_{\inf}(x_0, y_2, x_1, x_3) \wedge \varphi_{\sup}(x_0, x_4, y_1, y_2)\\ \leftrightarrow (\exists y)\ \varphi_{\sup}(x_0, y, x_2, x_3) \wedge \varphi_{\inf}(x_0, x_4, x_1, y)\end{aligned} \tag{5.85}$$

$$\begin{aligned}\mathsf{r}_u(x_1, x_0, x_2) \wedge \mathsf{r}_u(x_2, x_0, x_3)\\ \rightarrow (\exists y)\ \varphi_{\inf}(x_0, x_1, x_2, y) \wedge \varphi_{\sup}(x_0, x_3, x_2, y)\end{aligned} \tag{5.86}$$

$$\begin{aligned}(\exists y_1, y_2)\ \varphi_{\inf}(x_0, y_1, x_2, x_3) \wedge \varphi_{\sup}(x_0, y_2, x_2, x_3)\\ \wedge\, \mathsf{r}_u(y_1, x_0, x_1) \wedge \mathsf{r}_u(x_1, x_0, y_2)\\ \leftrightarrow \mathsf{r}_u(x_1, x_2, x_3)\end{aligned} \tag{5.87}$$

Let $\mathbf{2}'$ *be the structure* $(2;\ \mathsf{r}_u, \mathsf{r}_c)$*. A set* Σ *with Property* (5.33) *for* $\mathbf{2}'$ *is given by Equations* (5.79)–(5.87)*, and the following set of Equations* (5.88) *and* (5.89)*.*

$$\mathsf{r}_c(x, x') \wedge \varphi_{\inf}(x_0, x_1, x, x') \rightarrow x_0 \approx x_1 \tag{5.88}$$
$$\mathsf{r}_c(x, x') \wedge \varphi_{\sup}(x_0, x_1, x, x') \rightarrow \mathsf{r}_c(x_0, x_1) \tag{5.89}$$

Proof. Let $\mathbf{B}$ be a finite structure such that $\mathbf{B} \models \Sigma$. We need to show that $\mathbf{B} \in \mathrm{R\,P}\,\mathbf{2}$. We even show that $\mathbf{B} \in \mathrm{I\,P}\,\mathbf{2}$. The idea of the proof is to introduce an auxiliary Boolean algebra structure on B and derive the required representation of $\mathbf{B}$ from it. Since the Boolean relational clone O_9 does not contain $\leq^{\mathbf{2}}$ or $\mathsf{r}_i^{\mathbf{2}}$ or $\mathsf{r}_s^{\mathbf{2}}$, we can not expect to define a Boolean algebra structure by primitive-positive definitions. But the Boolean relational clone generated by r_u and r_0 is O_8, and we have $\leq^{\mathbf{2}}, \mathsf{r}_i^{\mathbf{2}}, \mathsf{r}_s^{\mathbf{2}} \in O_8$. Moreover, the dual of O_9 is O_9 itself. These observations suggest to fix an arbitrary zero element 0_B and to construct a Boolean algebra structure by primitive-positive definitions involving 0_B.

We denote an arbitrary but fixed element of B by 0_B, and define the relation $\leq$ on B by

$$b_1 \leq b_2 \quad \text{iff} \quad \mathsf{r}_u(b_1, 0_B, b_2).$$

By (5.79),(5.80), this defines an order relation. By (5.81)–(5.84), $(B, \leq)$ is a lattice order and for the infimum (supremum resp.) function inf (sup resp.) of this lattice we have

$$\begin{aligned} b_1 = \inf(b_2, b_3) \quad &\text{iff} \quad \varphi_{\inf}(0_B, b_1, b_2, b_3), \\ b_1 = \sup(b_2, b_3) \quad &\text{iff} \quad \varphi_{\sup}(0_B, b_1, b_2, b_3). \end{aligned}$$

By (5.85),(5.86), the lattice is distributive and complemented, that is, a Boolean algebra.

Let J be the set of all atoms of the Boolean algebra defined above. It is well known that the mapping $\alpha\colon B \to 2^J$ defined by

$$\alpha(b)(j) := \begin{cases} 1 & \text{if } j \leq b \\ 0 & \text{otherwise} \end{cases}, \qquad b \in B, j \in J,$$

is a bijection and an isomorphism with respect to the Boolean algebra structures defined on B and 2^J. By (5.87), r_u can be obtained by a primitive-positive definitions from $\varphi_{\inf}$, $\varphi_{\sup}$ and $\leq$. Hence, $\alpha\colon \mathbf{B} \to \mathbf{2}^J$ is an isomorphism, so $\mathbf{B} \in \mathrm{I}\,\mathrm{P}\,\mathbf{2}$.

To prove the assertion about $\mathbf{2}'$, we just observe that, by (5.88) and (5.89), it holds $\mathsf{r}_c(b, b')$ if and only if b and b' are complements in the Boolean algebra structure on B defined above. $\square$

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Erklärung gemäß § 5 (1) Punkt 5 a) und b) Promotionsordnung

Hiermit versichere ich, dass ich die vorliegende Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe; die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht. Die Arbeit wurde bisher weder im Inland noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt.

Die vorliegende Dissertation wurde an der Technischen Universität Dresden unter der Betreuung von Prof. Dr. Reinhard Pöschel angefertigt.

Affirmation

Hereby I affirm that I wrote the present thesis without any inadmissible help by a third party and without using any other means than indicated. Thoughts that were taken over directly or indirectly from other sources are indicated as such. This thesis has not been presented to any other examination board in this or a similar form, neither in this nor in any other country.

The present thesis has been written at the Dresden University of Technology under the supervision of Prof. Dr. Reinhard P"oschel.